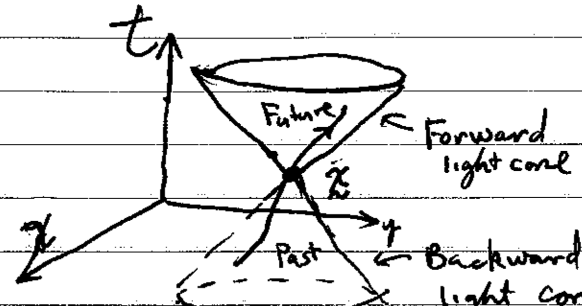
Lecture #23: Four-vectors an intro to space-time

Geometry of Space-time Diagrams

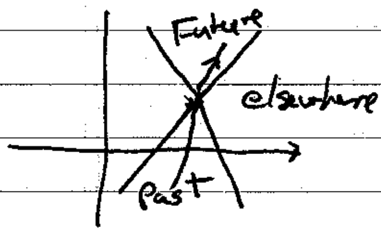
Event = geometric point in space-time
Coordinates (x^0, x^1, x^2, x^3) : Minkowski



World line – trajectory of a particle in space-time

Light cone – Surface of points traced out by a spherical light pulse eminating from (or converging on) a space-time point

Given two events: $|\Delta\vec{x}|^2 < (ct)^2$ Causually connected



$|\Delta\vec{x}|^2 > (ct)^2$ Not causually connected

$|\Delta\vec{x}|^2 = (ct)^2$ Connected by a light signal

• Invariant Interval $\boxed{c^2\Delta t^2 - |\Delta\vec{x}|^2 = \Delta S^2}$ Same in all reference frames

$\Delta S^2 \equiv$ effective length in Minkowski space

• $\Delta S^2 = 0 \Rightarrow$ Events are light-like separated

• $\Delta S^2 < 0 \Rightarrow$ Events are space-like separated

i.e. $\exists$ a frame in which $\Delta t = 0 \Rightarrow \Delta S^2 = |\Delta\vec{x}|^2$

(Not causally connected)

• $\Delta S^2 > 0 \Rightarrow$ Events are time-like separated

$\Rightarrow \exists$ a frame s.t. $\Delta x = 0 \Rightarrow \Delta t = \Delta\tau$ proper time

$\Rightarrow \Delta S^2 = c^2 \Delta\tau)^2$

Note: Given Δt for two events given particle moving with velocity u, $\Delta\tau = \frac{\Delta t}{\gamma(u)}$ is an invariant

Geometry of space time

- Recall 3D Euclidean geometry

Position vector $\vec{x} = (x^1, x^2, x^3)$

Under rotation to a new coordinate system:

$$x_i' = R_{ij}\, x_j \quad : \quad (\text{Rotation matrix } R_{ij})$$

(Einstein summation convention)

Length is preserved $x_i' x_i' = x_j x_j = |\vec{x}|^2$

Define a vector : transforms like position $V_i' = R_{ij} V_j$

- In Minkowski space

Position 4-vector $(x^0, x^1, x^2, x^3) = x^\mu$ (Contravariant coordinates)

Under Lorentz ~~tranformation~~:

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \quad \left(\text{Summation convention, repeated index, one lower one upper}\right)$$

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \quad \left(\text{"Boost" along } x^1\right)$$

Define a general Four-vector (contravariant) $V^\mu = (V^0, V^i_{\ i=1,2,3})$

Transforms like position

$$\boxed{V^{\mu\prime} = \Lambda^\mu{}_\nu V^\nu}$$

Lorentz Scalar field: $\psi'(x') = \psi(x)$ No change in value under Lorentz Transf

Metric

Under Lorentz Transformation the "distance" is preserved

$\Rightarrow$ Inner product $x \cdot x = (x^0)^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2)$

$$= (x^0)^2 - |\vec{x}|^2 = x' \cdot x'$$

Define metric tensor $x^\mu g_{\mu\nu} x^\nu = x \cdot x$

$$\Rightarrow g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

Defines the distance in Minkowski space

Define covariant coordinates $x_\mu = g_{\mu\nu} x^\nu = (x_0, -\vec{x})$

(Geometrically the coordinates w.r.t. to the dual basis)

Generally for any two four vectors

$V^\mu W_\mu$ = Lorentz scalar $\qquad W_\mu = g_{\mu\nu} W^\nu$

Also $g^{\mu\nu} = g_{\mu\nu} = g^{-1}_{\mu\nu} \Rightarrow W^\nu = g^{\nu\mu} W_\mu$ etc.

Example: Relativistic Doppler Shift

Consider a plane wave with wave vector $\vec{k}$, frequency ω

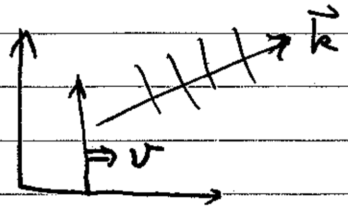
Define phase field $\phi(x) = \vec{k} \cdot \vec{x} - \omega t$

Physically, this must be a Lorentz scalar field

(Wave reaches its peak at a certain space-time point)

$$\Rightarrow \phi'(x') = \vec{k}' \cdot \vec{x}' - \omega' t' = \phi(x)$$

$$\Rightarrow \phi(x) = \vec{k}\cdot\vec{x} - \omega t = k_{\parallel} x_{\parallel} + \vec{k}_{\perp}\cdot\vec{x}_{\perp} - \frac{\omega}{c} x^0$$



· parallel component along v

· $\perp$ to v: $\vec{x}_{\perp}$

$$x_{\parallel} = \gamma(x'_{\parallel} + \beta x^{0\prime}) \qquad x^0 = \gamma(x^{0\prime} + \beta x'_{\parallel})$$
$$x_{\perp} = x'_{\perp}$$

$$\Rightarrow \phi(x) = k_{\parallel}\left(\gamma(x'_{\parallel} + \beta x^{0\prime})\right) + \vec{k}_{\perp}\cdot\vec{x}_{\perp} - \frac{\omega}{c}\left(\gamma(x^{0\prime} + \beta x'_{\parallel})\right)$$

$$= \gamma\left(k_{\parallel} - \beta\frac{\omega}{c}\right) x'_{\parallel} + \vec{k}_{\perp}\cdot\vec{x}'_{\perp} - \gamma\left(\frac{\omega}{c} - \beta k_{\parallel}\right) x'_0$$

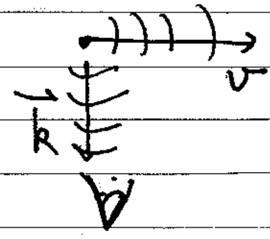
$$= \phi(x') = k'_{\parallel} x'_{\parallel} + \vec{k}'_{\perp}\cdot\vec{x}'_{\perp} - \frac{\omega'}{c} x_0$$

Thus $\boxed{k^{\mu} = (k^0, \vec{k}) \text{ with } k^0 = \frac{\omega}{c} \text{ is a four vector}}$

$$k^{0\prime} = \gamma(k^0 - \beta k_{\parallel}) \Rightarrow \boxed{\omega' = \gamma(\omega - v k_{\parallel})}$$
$$k'_{\parallel} = \gamma(k_{\parallel} - \beta k_0) \Rightarrow \boxed{k'_{\parallel} = \gamma\left(k_{\parallel} - \frac{v}{c^2}\omega\right)}$$
$$\vec{k}'_{\perp} = \vec{k}_{\perp}$$

$\beta \ll 1 \quad \gamma \sim 1 \quad \omega' = \omega - k_{\parallel} v$

<u>Note</u>: Even for ~~waves~~ emitter moving $\perp$ to observer there is a Doppler shift in ω



$k_{\parallel} = 0$

$|\vec{k}| = \frac{\omega}{c}$

$$\omega' = \gamma\omega \approx \left(1 - \frac{v^2}{2c^2}\right)\omega$$

quadratic

Generally $\tan\theta = \frac{|\vec{k}_{\perp}|}{k_{\parallel}}$ changes

Other four vectors

- Proper velocity: First note that $\frac{dx^\mu}{dt}$ is not a four-vector since t transforms under Lorentz. However if we use the proper time (i.e. time in the particles own rest frame): $d\tau$

$$u^\nu = \frac{dx^\nu}{d\tau}, \quad \text{Now } d\tau = \frac{dt}{\gamma(u)}$$

where u = speed in Lab frame

$$\Rightarrow \boxed{u^\nu = \gamma(u)\frac{dx^\nu}{dt} = (\gamma(u)c,\ \gamma(u)\vec{u})}$$

- ~~Four momentum~~ Relativistic invariant

$$u^\nu u_\nu = (\gamma(u))^2 c^2 - \gamma(u)^2 u^2 = \gamma(u)^2(c^2 - u^2)$$

$$= \frac{1}{1-\frac{u^2}{c^2}}(c^2 - u^2) = c^2 \quad \checkmark$$

- ~~Proper~~ Four-momentum

$$p^\nu = \underset{\substack{\uparrow \\ \text{rest mass}}}{m} u^\nu = \underbrace{\gamma(u) m}_{\substack{\uparrow \\ \text{relativistic mass}}} \frac{dx^\nu}{dt} = (\gamma(u) m c,\ \gamma(u) m \vec{u})$$

Empirical fact: All components of p^ν are conserved

$$p^0 = \gamma(u) mc = E/c \quad \text{relativistic energy}/c$$

Limit $u \ll c$ $\quad \gamma(u) \approx 1 + \frac{u^2}{2c^2} \Rightarrow cp^0 = \underset{\substack{\uparrow \\ \text{rest energy}}}{mc^2} + \frac{1}{2}mu^2$

Relativistic invariant $\quad p^\nu p_\nu = (mc)^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$

- **Four-current density**

Charge conservation: True in all reference frames

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}'\cdot\vec{J}' + \frac{\partial \rho'}{\partial t'} = 0$$

$$\parallel$$

$$\frac{\partial}{\partial x_\parallel} J_\parallel + \vec{\nabla}_\perp\cdot\vec{J}_\perp + \frac{\partial \rho}{\partial t} = 0$$

Chain rule

$$\begin{cases} \dfrac{\partial}{\partial x_\parallel} = \dfrac{\partial x'_\parallel}{\partial x_\parallel}\dfrac{\partial}{\partial x'_\parallel} + \dfrac{\partial t'}{\partial x_\parallel}\dfrac{\partial}{\partial t'} = \gamma\left(\dfrac{\partial}{\partial x'_\parallel} - \dfrac{v}{c^2}\dfrac{\partial}{\partial t'}\right) \\ \dfrac{\partial}{\partial t} = \gamma\left(\dfrac{\partial}{\partial t'} - v\dfrac{\partial}{\partial x'_\parallel}\right) \\ \vec{\nabla}'_\perp = \vec{\nabla}_\perp \end{cases}$$

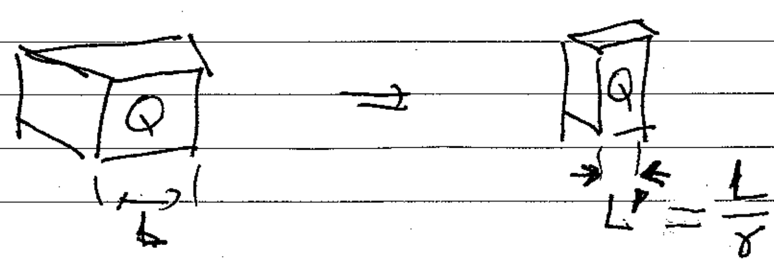
$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = \gamma\left(\frac{\partial}{\partial x'_\parallel} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)J_\parallel + \vec{\nabla}'_\perp\cdot\vec{J}_\perp + \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'_\parallel}\right)\rho$$

$$= \frac{\partial}{\partial x'_\parallel}\left[\gamma(J_\parallel - v\rho)\right] + \vec{\nabla}'_\perp\cdot\vec{J} + \frac{\partial}{\partial t'}\left[\gamma\left(\rho - \frac{v}{c^2}J_\parallel\right)\right]$$

$\therefore$ $\boxed{J^\mu = (c\rho, \vec{J}) \text{ is a Four vector}}$

$\vec{J} = 0$ in S frame $\Rightarrow$ $\rho' = \gamma\rho$ is S'

Physically: Length contraction



Length contraction