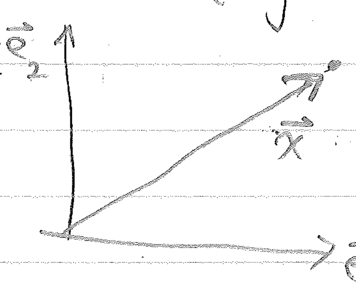
Lecture #25: Non-Euclidean Space-Time Geometry

Space and time together form a 4-dimensional geometry, but not the familiar Euclidean geometry of 3-dimensional space. This is reflected in the fact that the interval $\Delta S^2 = (c\Delta t)^2 - (\Delta \vec{x})^2$ is Lorentz invariant whereas the invariant length in a Euclidean geometry is $R^2 = x^2 + y^2 + z^2$

- Brief overview of non-Euclidean geometry

- Euclidean geometry: $\mathbb{R}^n$



A geometric point in $\mathbb{R}^n$ in a vector $\vec{x}$, which can be represented as coördinates w.r.t. a basis.

$$\vec{x} = x^i \vec{e}_i$$ (Einstein summation convention: sum upper w/ lower index)

The length of the vector (distance from the origin)

$$|\vec{x}| = \sqrt{(x^1)^2 + (x^2)^2 + \dots + (x^n)^2}$$

↑ Euclidean distance via Pythagoras

Endowing a vector space with a distance is equivalent to defining an "inner product" (or dot product).

$$|\vec{x}|^2 \equiv \vec{x} \cdot \vec{x}$$

Given a basis $\{\vec{e}_i\}$

$$|\vec{x}|^2 = (x^i \vec{e}_i) \cdot (x^j \vec{e}_j) = x^i x^j \vec{e}_i \cdot \vec{e}_j$$

$$\boxed{\vec{e}_i \cdot \vec{e}_j \equiv g_{ij} = \text{The } \underline{\text{metric}} \text{ of the space}}$$

$$|\vec{x}|^2 = x^i g_{ij} x^j$$

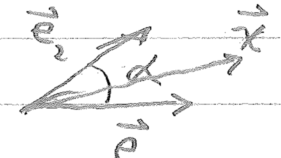
For Euclidean geometry $g_{ij} = \delta_{ij}$

$\Rightarrow \quad |\vec{x}|^2 = x^i x^i$ (as before)

- <u>Non-Euclidean geometry</u>

Suppose we decompose $\vec{x}$ w.r.t. a basis for which $\vec{e}_i \cdot \vec{e}_j \neq \delta_{ij} \Rightarrow$ non-Euclidean metric

E.g



$$\vec{x} = x^1 \vec{e}_1 + x^2 \vec{e}_2$$

$$|\vec{x}|^2 = (x^1)^2 + (x^2)^2 + 2(x^1 x^2) \cos\alpha$$

(Law of cosines)

Metric $\quad \underset{\sim}{g} = \begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix}$

$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = 1$

$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_1 = \cos\alpha$

$$|\vec{x}|^2 = [x^1 \; x^2] \begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

Transformations that leave length invariant

Consider a change of basis matrix $\underset{\approx}{\Lambda}$

$$\vec{e}_i' = \Gamma_i{}^j \vec{e}_j$$

New coordinates $\vec{x} = x'^i \vec{e}_i'$

$$|\vec{x}|^2 = x^i \underbrace{\vec{e}_i \cdot \vec{e}_j}_{g_{ij}} x^j = x'^i \underbrace{\vec{e}_i' \cdot \vec{e}_j'}_{g'_{ij}} x'^j$$

$$g'_{ij} = (\Gamma_i{}^k \vec{e}_k) \cdot (\Gamma_j{}^l \vec{e}_l) = \Gamma_i{}^k g_{kl} \Gamma_j{}^l$$

$$\Rightarrow \boxed{\underset{\approx}{g}' = \underset{\approx}{\Gamma}\, \underset{\approx}{g}\, \underset{\approx}{\Gamma}^T}$$

A transformation that leaves the length invariant w.r.t new coordinates $\boxed{\underset{\approx}{g} = \underset{\approx}{\Gamma}\, \underset{\approx}{g}\, \underset{\approx}{\Gamma}^T} \Rightarrow$ Preserves the metric

For Euclidean $g_{ij} = \delta_{ij} \Rightarrow \underset{\approx}{g} = \underset{\approx}{\mathbb{1}}$

$$\Rightarrow \underset{\approx}{\Gamma}\, \underset{\approx}{\Gamma}^T = \underset{\approx}{\mathbb{1}} \quad \text{orthogonal transformation}$$

$$\underset{\approx}{\Gamma}^T = \underset{\approx}{\Gamma}^{-1}, \quad \text{e.g. rotation/reflection}$$

Define $\underset{\approx}{\Lambda} = (\underset{\approx}{\Gamma}^T)^{-1}$

For Euclidean geometry $\underset{\approx}{\Lambda} = \underset{\approx}{\Gamma}$

Transformation on coordinates

Two representations: $\vec{x} = x^i \vec{e}_i = x'^j \vec{e}_j{}'$

new coords   new basis

$$\Rightarrow x^i \vec{e}_i = x'^j (\Gamma_j{}^i \vec{e}_i)$$

$$\Rightarrow \underset{\sim}{x} = \underset{\sim}{x}' \underset{\approx}{\Gamma} = \underset{\approx}{\Gamma}^T \underset{\sim}{x}'$$

$$\underset{\sim}{x}' = (\underset{\approx}{\Gamma}^T)^{-1} \underset{\sim}{x} = \underset{\approx}{\Lambda} \underset{\sim}{x}$$

$$\boxed{x'^j = \Lambda^j{}_i x^i}$$

For Euclidean geometry coordinates transform as basis vectors, but not true for non-Euclidean case.

Also note: If transformation <u>preserves length</u>

$$\underset{\approx}{\Gamma}^{-1} \underset{\approx}{g} (\Gamma^T)^{-1} = \underset{\approx}{g}$$

$$\Rightarrow \boxed{\underset{\approx}{\Lambda}^T g \underset{\approx}{\Lambda} = \underset{\approx}{g}}$$ Another way of preserving metric

<u>Aside</u>: $\boxed{\Lambda^j{}_i = \frac{\partial x'^j}{\partial x^i}}$

$$\boxed{\Gamma_i{}^j = (\Lambda^i{}_j)^{-1} = \frac{\partial x^j}{\partial x'^i}}$$

Dual Basis: Covariant & Contravariant Coords

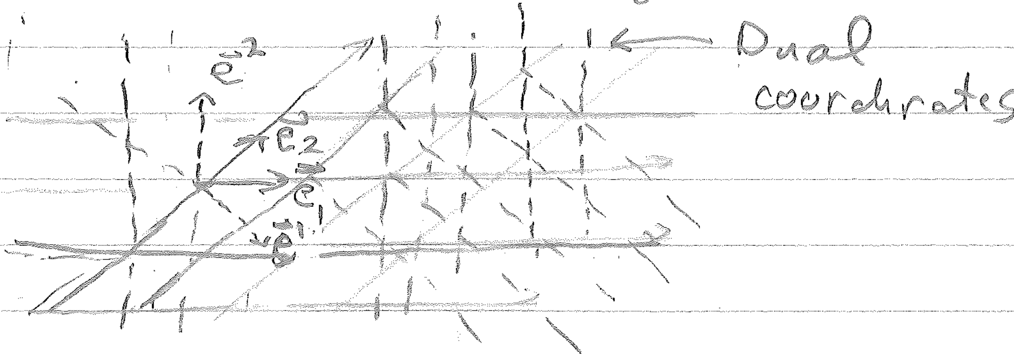
Given a basis in a general non-Eucledian geometry, $\{\vec{e}_i\}$, we define a dual basis $\{\vec{e}^{\,i}\}$ so that $\boxed{\vec{e}_i \cdot \vec{e}^{\,j} = \delta_i^{\ j}}$

Bi-orthogonal

For a Euclidean space, the basis is it's own dual $\vec{e}_x = \vec{e}^{\,x}$: No distinction.

The dual basis spans the space $\Rightarrow$ Dual coordinates

$$\vec{x} = x^i \vec{e}_i = x_i \vec{e}^{\,i}$$



Transformation Law: New basis $\vec{e}_i'$

New dual $\vec{e}'^{\,i}$

$\vec{e}_i' \, \vec{e}^{\,j} = \delta_i^{\ j}$ and $\vec{e}_i' = \Gamma_i^{\ j} \vec{e}_j$

$$\Rightarrow \boxed{\vec{e}'^{\,i} = (\Gamma_i^{\ j})^{-1\,T} \vec{e}^{\,j} = \Lambda^i_{\ j} \vec{e}^{\,j}}$$

New coordinates $\boxed{x_i' = \Gamma_i^{\ j} x_j}$

⇒ Coordinates x_i w.r.t. dual basis $\vec{e}^{\,i}$ transform in the the same way as the basis vectors $\vec{e}_i$

⇒ $\{x_i\}$ as said to be <u>covariant</u> coordinates

Coordinates w.r.t. basis $\vec{e}_i$, $\{x^i\}$ transform oppositely, as dual vectors $\vec{e}^{\,i}$

⇒ $\{x^i\}$ are said to be <u>contravariant</u> coordinates

Transformation laws:

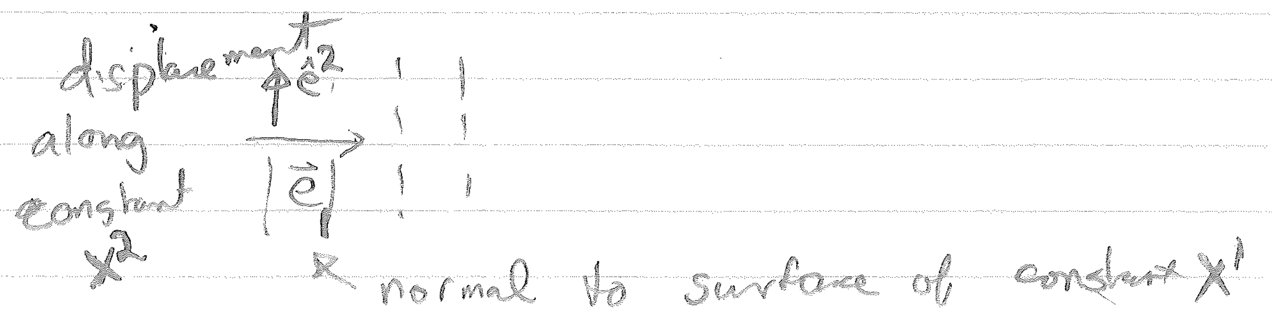
$$x'^j = \Lambda^j{}_i \, x^i = \frac{\partial x'^j}{\partial x^i} \, x^i$$

$$x'_j = \Gamma_j{}^i \, x_i = \frac{\partial x^i}{\partial x'^j} \, x_i$$

<u>Geometric interpretation</u>

There are two means of vectors:

Diplacements and normal to level surfaces



dx^i = displacement that changes x^i (normal)

$\Rightarrow$ Contravariant coordinates

A displace along x^i generated by the derivative $\frac{\partial}{\partial x^i}$

Note $\boxed{\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = \Gamma_i^{\ j} \frac{\partial}{\partial x^j}}$

The component of the gradient w.r.t contravariant coordinates are co-variant coords

Denote $\partial_i = \frac{\partial}{\partial x^i}$ (upper index in denominator = lower index)

Relationship between contravariant and covariant coordinates:

$$\vec{X} = x_i \vec{e}^{\,i} = x^i \vec{e}_i$$

$$x_i \underbrace{\vec{e}^{\,i} \cdot \vec{e}_j}_{\delta_i^j} = x^i \underbrace{\vec{e}_i \cdot \vec{e}_j}_{g_{ij}}$$

$$\Rightarrow \boxed{x_j = x^i g_{ij} = g_{ji} x^i} \quad \text{symmetric}$$

$\Rightarrow$ Metric converts between covariant + contravariant

Similarly $g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j}$ $\Rightarrow$ $\boxed{x^i = g^{ij} x_j}$

Minkowski Space

Greek letters 0, 1, 2, 3

Space time coordinates: $x^{\mu} = (x^0, x^1, x^2, x^3)$

$x^0 = ct$

$x^1 = x$

$x^2 = y$

$x^3 = z$

$= (x^0, x^i)$ Latin letter $i = 1, 2, 3$

$= (x^0, \vec{x})$

Lorentz invariant interval:

$$(ds)^2 = (dx^0)^2 - |d\vec{x}|^2 = dx^{\mu} g_{\mu\nu} dx^{\nu}$$

Metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

Contravariant coordinate transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \quad \Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

Lorentz invariance

$$\Lambda^{\alpha}{}_{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = g_{\mu\nu}$$

$$\underset{\approx}{\Lambda}^T \underset{\approx}{g} \underset{\approx}{\Lambda} = \underset{\approx}{g}$$

Boost with v along x:

$$\underset{\approx}{\Lambda} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Covariant $x_{\mu} = g_{\mu\nu} x^{\nu}$

$$\Rightarrow \quad x'_{\mu} = g_{\mu\nu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}{}_{\lambda} x^{\lambda} = g_{\mu\nu} \Lambda^{\mu}{}_{\lambda} g^{\lambda\delta} x_{\delta}$$

$$\Rightarrow x'_\mu = \Lambda_\mu{}^\nu x_\nu$$

$$\Rightarrow \Gamma_\mu{}^\nu = \left(\Lambda^\mu{}_\nu\right)^{-1\,T} = \Lambda_\mu{}^\nu$$

$$\boxed{\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}} \qquad \boxed{\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x'_\mu}{\partial x_\nu}}$$

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\lambda = \delta_\nu{}^\lambda$$

$$\underset{\approx}{\Lambda}\, \underset{\approx}{\Lambda}^{-1} = \mathbb{1}$$

<u>For Minkowski space</u>

$$V^\mu = (V^0, \vec{V})$$

$$\Rightarrow V_\mu = (V^0, -\vec{V}) \quad \Rightarrow \begin{cases} V_0 = V^0 \\ V_i = -V^i \end{cases}$$

Inner product: $\|V\|^2 = V^\mu V_\mu = V^\mu g_{\mu\nu} V^\nu$

(Lorentz invariant) $= (V^0)^2 - |\vec{V}|^2$

Example: $x^\mu = (ct, \vec{x})$

$$x_\mu = (ct, -\vec{x})$$

$$p^\mu = m\frac{dx^\mu}{d\tau} = \left(\frac{E}{c}, \vec{p}\right)$$

$$p_\mu = \left(\frac{E}{c}, -\vec{p}\right)$$

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$$

$$J^\mu = (c\rho, \vec{J})$$
$$J_\mu = (c\rho, -\vec{J})$$

Note: Gradient in Minkowski space

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad \text{Covariant coordinates}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$
$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$\Rightarrow \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = -(\text{d'Alembertian})$$

Lorentz invariant!

Tensors: $T^{\mu\nu}$

$$\boxed{T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}}$$

Example: Outer product: $T^{\mu\nu} = A^\mu B^\nu$

Raising and lowering indecies $T_\mu{}^\nu = g_{\mu\alpha} T^{\alpha\nu}$

$$T_0{}^0 = g_{0\alpha} T^{\alpha 0} = g_{00} T^{00} = T^{00}$$

$$T_0{}^i = g_{0\alpha} T^{\alpha i} = g_{00} T^{0i} = T^{0i}$$

$$T_i{}^0 = g_{i\alpha} T^{\alpha 0} = g_{ii} T^{i0} = -T^{i0}$$

$$T_i{}^j = g_{ik} T^{kj} = -T^{ij}$$

Generally: $T^{\alpha\beta} = \left[\begin{array}{c|c} t\text{-}t & t\text{-}s \\ \hline s\text{-}t & s\text{-}s \end{array}\right]$

$$T_\alpha{}^\beta = \left[\begin{array}{c|c} + & + \\ \hline - & - \end{array}\right], \quad T^\alpha{}_\beta = \left[\begin{array}{c|c} + & - \\ \hline + & - \end{array}\right]$$

$$T_{\alpha\beta} = \left[\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right]$$