Lecture 26: The Manefestly Covariant Electromagnetism

We know that Maxwell's Equations are the same in all reference frames, but in the standard 3-vectors $(\vec{E}, \vec{B})$ plus time, they are not "manefestly covariant". By this, we mean

$$\left(\text{LHS}: \overset{\text{Minkowski}}{\text{Tensor}} \text{ of rank } k\right) = \left(\text{RHS} = \overset{\text{Minkowski}}{\text{Tensor}} \text{ of rank } k\right)$$

For example: $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ is not manefestly covariant, ρ is not a Lorentz scaler, for under a boost $\rho' = \gamma\left(\rho - \frac{v}{c^2} J_{\parallel}\right)$.

We don't know how $\vec{\nabla} \cdot \vec{E}$ transforms under a Lorentz transformation, though it must do so so that $\nabla' \cdot \vec{E}' = 4\pi\rho'$. So Gauss's Law is covariant, it's just not manefestly covariant.

Potential Formulation

To express electromagnetism in a manefestly covariant form, we begin with the potential formulation.

In the "Lorentz Gauge" $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

we have

$$\Box \phi = -4\pi\rho \quad , \quad \Box \vec{A} = -\frac{4\pi}{c}\vec{J}$$

where $\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

We found that $J^\mu = (c\rho, \vec{J})$ is a 4-vector.
And $\partial_\mu \partial^\mu = -\Box$ is a Lorentz scalar

$\Rightarrow$ We define the 4-potential (4-vector)

$$A^\mu = (\phi, \vec{A})$$

whose equation of motion is

$$\boxed{\Box A^\mu = -\frac{4\pi}{c} J^\mu}$$

Manifestly covariant!

This is true in all reference frames because the Gauge choice is the same in all frame.

Lorentz Gauge: $\boxed{\underbrace{\partial_\mu A^\mu}_{\text{Lorentz invariant}} = \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0}$

Note: If we take the 4-divergence of the wave-eqn.

$$\partial_\mu (\Box A^\mu) = \Box(\underbrace{\partial_\mu A^\mu}_{=0}) = -\frac{4\pi}{c}\partial_\mu J^\mu$$

$$\Rightarrow \partial_\mu J^\mu = 0 \Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

Charge conservation contained in Maxwell's eqns as it must!

Electromagnetic Field Tensor

How are the electric and magnetic fields expressed in Minkowski space?

We have $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$, $\quad \vec{B} = \vec{\nabla} \times \vec{A}$

Remember that a cross-product is intimately related to a 2^{nd} rank anti-symmetric tensor

$$B_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k$$

$$B_{ij} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix}$$

Consider, thus, the "Four-Curl" Component

Define $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$ Electromag field tensor

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{1}{c}\frac{\partial}{\partial t} A^i - (-\partial_i \phi)$$

$$= \left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi\right)_i = -\vec{E}_i = -F^{i0}$$

$$F^{ij} = -F^{ji} = \partial^i A^j - \partial^j A^i = -(\partial_i \vec{A}_j - \partial_j \vec{A}_i$$

$$= -B_{ij} = -\epsilon_{ijk} \vec{B}_k$$

$\Rightarrow$ $F^{\mu\nu}$ = Electromagnetic Field Tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

<u>Maxwell's Equations</u> (from sources)

<u>Four-div:</u>

$$\partial_\mu F^{\mu\nu} = \underbrace{\partial_\mu \partial^\mu}_{-\Box} A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{= \partial^\nu(\partial_\mu A^\mu) = 0}$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

$\Leftarrow$ Manifestly covariant!

(LHS: 4 vector) = (RHS: 4-vec

Components:

$$\frac{1}{c}\frac{\partial}{\partial t} F^{0\nu} + \partial_i F^{i\nu} = \frac{4\pi}{c} J^\nu$$

- $\nu = 0 \Rightarrow \partial_i F^{i0} = \frac{4\pi}{c} J^0 \Rightarrow \partial_i E_i = \frac{4\pi}{c}(c\rho)$

$$\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$$ Gauss's Law

- $\nu = j \Rightarrow \frac{1}{c}\frac{\partial}{\partial t} F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c} J^j$

$$\frac{1}{c}\frac{\partial}{\partial t}(-\vec{E}_j) + \partial_i(-\epsilon_{ijk}\vec{B}_k) = \frac{4\pi}{c}\vec{J}_j$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$$ Ampere's Law

What about the other Maxwell's equations?

They follow from the definition of $F^{\mu\nu}$ in terms of the potentials

3D

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Consider, thus, the "4-curl" of $F^{\mu\nu}$

$$\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = \epsilon^{\mu\nu\lambda\sigma}\partial_\nu\left(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda\right)$$

$$= \epsilon^{\mu\nu\lambda\sigma}\partial_\nu \partial_\lambda A_\sigma - \underbrace{\epsilon^{\mu\nu\lambda\sigma}\partial_\nu \partial_\sigma A_\lambda}_{\epsilon^{\mu\nu\sigma\lambda}\partial_\nu \partial_\lambda A_\sigma}$$

$$= 2\underbrace{\epsilon^{\mu\nu\lambda\sigma}}_{\substack{\text{Antisymmetric} \\ \mu \leftrightarrow \lambda}}\underbrace{\partial_\mu \partial_\lambda}_{\substack{\text{symmetric} \\ \mu \leftrightarrow \lambda}} A_\sigma$$

$$\Rightarrow \boxed{\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0}$$

Note: Sometimes this is writed in terms of the "dual"

$$\tilde{\mathcal{F}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & +E_z & -E_y \\ B_y & -E_z & 0 & +E_x \\ B_z & +E_y & -E_x & 0 \end{bmatrix}$$

$$\boxed{\partial_\nu \tilde{\mathcal{F}}^{\mu\nu} = 0}$$

Switch $\vec{E}$ & $\vec{B}$

Check

$\mu=0: \quad \partial_\nu \tilde{F}^{0\nu} = \partial_i \tilde{F}^{0i} = \partial_i(-\vec{B}_i) = 0$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B}=0} \quad \checkmark$$

$\mu=i \quad \partial_\nu \tilde{F}^{i\nu} = \partial_0 \tilde{F}^{i0} + \partial_j \tilde{F}^{ij}$

$$= \frac{1}{c}\frac{\partial}{\partial t}(\vec{B}_i) + \partial_j\,(\epsilon_{ijk}\vec{E}_k)$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}} \quad \checkmark$$

Lorentz Transformation on fields

Lorentz boost: along x $\qquad F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$

$$F'^{10} = E'_x = E_x$$

$$F'^{20} = E'_y = \gamma(E_y - \beta B_z)$$

$$F'^{30} = E'_z = \gamma(E_z - \beta B_y)$$

$$F'^{32} = B'_x = B_x$$

$$F'^{13} = B'_y = \gamma(B_y + \beta E_z)$$

$$F'^{21} = B'_z = \gamma(B_z - \beta E_z)$$

Generally: $\quad \vec{E}'_{\parallel} = \vec{E}_{\parallel} \quad , \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \frac{\vec{v}}{c}\times\vec{B}_\perp) \;,\; \vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{\vec{v}}{c}\times\vec{E}_\perp)$$

Finally, we seek a manefestly covariant form of the Lorentz force law

3D + time: $\vec{f} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

We have at our disposal:

Minkowski force: $K^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \vec{f}\right)$

Four velocity: $u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{u})$

E-M Field tensor: $F^{\mu\nu}$

Guess: $\boxed{K^\mu = q\, F^{\mu\nu} \frac{u_\nu}{c}}$

Manefestly covariant: Both sides are 4-vectors (Note, the sign is not obvious - empirical)

Check: $K^i = q\left(F^{i0}\frac{u_0}{c} + F^{ij}\frac{u_j}{c}\right)$

$$\gamma \vec{f}_i = q\,\vec{E}_i\left(\frac{\gamma c}{c}\right) + q\left(-\epsilon_{ijk}\vec{B}_k\right)\left(\frac{-\gamma \vec{u}_j}{c}\right)$$

$$\Rightarrow \boxed{\vec{f} = q\vec{E} + q\frac{\vec{u}}{c}\times\vec{B}}$$

The $\mu = 0$ component $K^0 = q\, F^{0i}\frac{u_i}{c}$

$$\Rightarrow \frac{\gamma}{c}\frac{dE}{dt} = q(-E_i)\left(\frac{-\gamma \vec{u}_i}{c}\right) \Rightarrow \boxed{\frac{d}{dt}E = q\vec{v}\cdot\vec{E}}$$

rate at which work is done on charge by field

In Summary, Electromagnetism is specified by three equations:

$$\boxed{\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ K^\mu &= q\, F^{\mu\nu} \frac{u_\nu}{c} \end{aligned}}$$

The End!