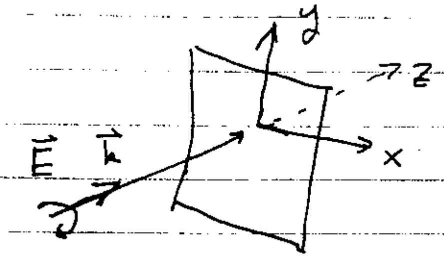
Physics 511

Problem Set # 7
Solutions

(1) Angular momentum in electromagnetic waves



Right-Hand circularly polarized wave travelling along $\hat{z}$

$$\vec{E}(z,t) = E_0\left(\hat{x}\cos(kz-\omega t) - \hat{y}\sin(kz-\omega t)\right)$$

At $z=0$

$$\vec{E}(0,t) = E_0\left(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right)$$

Equation of motion of a particle w/ mass m, charge q as an isotropic damped SHO, driven by the field

$$\frac{d^2\vec{r}}{dt^2} + \Gamma\frac{d\vec{r}}{dt} + \omega_0^2\vec{r} = \frac{q}{m}\vec{E}(0,t)$$

Use complex amplitudes: $\vec{E}(t) = \mathrm{Re}(\vec{\underset{\sim}{E}}\, e^{-i\omega t})$

$$\Rightarrow \vec{\underset{\sim}{E}} = E_0(\hat{x} + i\hat{y})$$

$$\vec{r} = \mathrm{Re}(\vec{\underset{\sim}{r}}\, e^{-i\omega t})$$

$$(-\omega^2 - i\omega\Gamma + \omega_0^2)\,\vec{\underset{\sim}{r}} = \frac{q}{m}\vec{\underset{\sim}{E}}$$

$$\vec{\underset{\sim}{r}} = \frac{q/m}{\omega_0^2-\omega^2-i\omega\Gamma}\,\vec{\underset{\sim}{E}} \equiv \alpha\,\vec{\underset{\sim}{E}}$$

Steady state motion $\vec{r}(t) = \mathrm{Re}(\vec{\underset{\sim}{r}}\, e^{-i\omega t})$

(a) Time average torque on charges

$$\langle \vec{\tau} \rangle = \langle \vec{r}(t) \times \vec{F}(t) \rangle = q \langle \vec{r}(t) \times \vec{E}(t) \rangle$$

Using our favorite rule: $\langle A(t)\, B(t) \rangle = \frac{1}{2} Re\left(\underset{\sim}{A}\, \underset{\sim}{B}^* \right)$

$$\Rightarrow \langle \vec{\tau} \rangle = \frac{q}{2} Re\left(\underset{\sim}{\vec{r}} \times \underset{\sim}{\vec{E}}^* \right) = \frac{q}{2} Re\left(\alpha\, \underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{E}}^* \right)$$

$$= \frac{q E_0^2}{2} Re\left(\alpha\, (\hat{x} + i\hat{y}) \times (\hat{x} - i\hat{y}) \right) = \frac{q E_0^2}{2} Re\left(-2i\alpha\, \hat{z} \right)$$

$$\langle \vec{\tau} \rangle = q E_0^2\, Re(-i\alpha)\, \hat{z} = q E_0^2\, Im(\alpha)\, \hat{z}$$

Thus the time average torque on the absorbing charges is:

$$\boxed{\langle \vec{\tau} \rangle = q E_0^2\, Im(\alpha)\, \hat{z}\,, \quad \alpha = \frac{q/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}}$$

(b) Rate at which fields do work on charges

$$\text{Power} = \frac{dW}{dt} = \vec{v} \cdot \vec{F} = \frac{d\vec{r}}{dt} \cdot q\vec{E}$$

$$\Rightarrow \frac{d}{dt} \langle W \rangle = \langle \vec{v} \cdot \vec{F} \rangle = \frac{1}{2} Re\left(-i\omega\, \underset{\sim}{\vec{r}} \cdot q\, \underset{\sim}{\vec{E}}^* \right)$$

$$= \omega \frac{q}{2} Re\left(-i \underset{\sim}{\vec{r}} \cdot \underset{\sim}{\vec{E}}^* \right) = \omega \frac{q}{2} Re\left(-i\alpha\, \underset{\sim}{\vec{E}} \cdot \underset{\sim}{\vec{E}}^* \right)$$

$$= \omega \frac{q E_0^2}{2} Re\left(-i\alpha\, (\hat{x} + i\hat{y}) \cdot (\hat{x} - i\hat{y}) \right) = \omega \frac{q}{2} E_0^2\, Re(-2i\alpha)$$

$$\Rightarrow \boxed{\frac{d}{dt} \langle W \rangle = \omega\, q E_0^2\, Im(\alpha)}$$

(c) From parts (a) and (b) we have shown that

$$\langle \vec{\tau} \rangle = \frac{1}{\omega} \frac{d}{dt} \langle W \rangle \ \hat{z}$$

The torque is the rate of change of angular momentum

$$\Rightarrow \langle \vec{\tau} \rangle = \frac{d}{dt} \langle \vec{L} \rangle$$

$$\therefore \boxed{\langle \vec{L} \rangle = \frac{\langle W \rangle}{\omega} \hat{z} \quad (\text{here } \hat{k} = \text{direction of prop.} = \hat{z})}$$

Here we have related the angular momentum induced in the charge to the energy of the charges. Where do these quantities come from?

Answer: From the field!

Thus we interpret $\langle \vec{L} \rangle$ as the angular momentum in the circularly polarized wave.

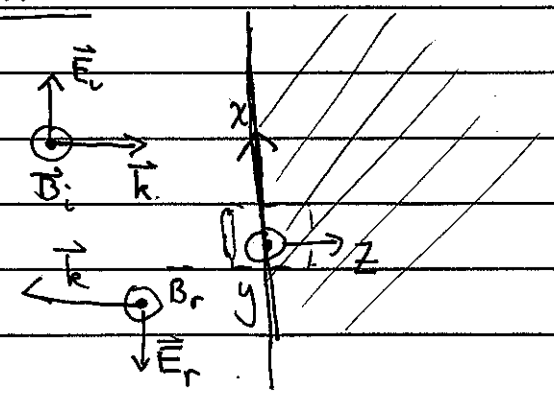
Since, in the quantum theory, the energy of a photon is $\hbar\omega$, the angular momentum of a RHC photon ~~is~~ $\hbar$ in the direction of propagation

Generally, any elementary particle has "intrinsic" angular momentum known as spin, whose value is $S\hbar$ where S is either an integer or half integer.

If S = integer particle is a "Boson"
$S = \frac{1}{2}$ - integer " " a "Fermion"

The spin of a photon is $S = 1$, so photons are Bosons.

Problem 2



$\vec{E} = \vec{B} = 0$
inside

(a) The force transmitted by the electromagnetic fields into the "pill box" is $\vec{F} = \oint_{\text{(surface of pill box)}} \overleftrightarrow{T} \cdot \hat{n}\, da$

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left(\vec{E}\vec{E} + \vec{B}\vec{B} + \frac{1}{2}\overleftrightarrow{\mathbb{1}}(E^2+B^2)\right)$$

- On the cap for $z<0$ $\quad \hat{n} = -\vec{e}_z, \quad \vec{E}\cdot\vec{e}_z = \vec{B}\cdot\vec{e}_z = 0$

$$\Rightarrow -\overleftrightarrow{T}\cdot\hat{n}\,da = \frac{1}{8\pi}(E_<^2 + B_<^2)\,da\ \vec{e}_z$$

- On the cap for $z>0$ $\quad \vec{E} = \vec{B} = 0 \Rightarrow \overleftrightarrow{T} = 0$
- On the lateral surface of the cylinder

$\hat{n} = \vec{e}_\rho = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y \Rightarrow \int_{\text{lateral surface}} (-\overleftrightarrow{T}\cdot\hat{n})\,da = 0$ since $\int\cos\phi\,d\phi = \int\sin\phi\,d\phi = 0$

$$\therefore \oint(-\overleftrightarrow{T}\cdot\hat{n})\,da = \left(\frac{1}{8\pi}(E^2+B^2)\,\vec{e}_z\right)\underbrace{\left(\text{Area of cap}\right)}_{(\equiv A)} = \vec{F}$$

$$\Rightarrow \text{Pressure} = \frac{\langle|\vec{F}|\rangle}{A} = \frac{1}{8\pi}\langle E_<^2 + B_<^2\rangle$$

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Now for $z<0$ $\cancel{\vec{E}_{total} = \vec{E}_r + \vec{E}_i}$

$$\vec{E}_<(0,t) = \vec{E}_i(0,t) + \vec{E}_r(0,t) = 0$$

$$\vec{B}_<(0,t) = \vec{B}_i(0,t) + \vec{B}_r(0,t) = 2\vec{B}_i \cos(\omega t - kz)$$

$$= 2E_0 \cos(\omega t - kz)$$

$$\Rightarrow \text{Pressure} = \frac{E_0^2}{2\pi}\langle \cos^2(\omega t - kz)\rangle = \boxed{\frac{E_0^2}{4\pi}}$$

(b) We also know that the force on a finite volume with zero charge density, and non-zero current is

$$\vec{F} = \int \frac{1}{c}(\vec{J}\times\vec{B})\, d^3x$$

The surface current at $z=0$ must agree with the general boundary condition: $|\Delta \vec{B}_{\parallel}| = \frac{4\pi}{c}K$

$$\Rightarrow |\vec{B}_{outside} - \vec{B}_{inside}| = \frac{4\pi}{c}K$$

$$\Rightarrow 2E_0\cos\omega t = \frac{4\pi}{c}K \Rightarrow \vec{K} = \frac{c}{2\pi}E_0\cos\omega t\, \vec{e}_x$$

We get the sign and direction of $\vec{K}$ from R.H.R

$$\Rightarrow \vec{J} = \vec{K}\delta(z) = \frac{c}{2\pi}E_0\cos\omega t\,\delta(z)\,\vec{e}_x$$

Since we want to disregard the self-force of current sheet on itself, we consider only the effect of incident wave on the current

$$\Rightarrow \text{Pressure} = \frac{\langle F\rangle}{A} = \frac{1}{A}\int\left\langle \frac{\vec{J}\times\vec{B}}{c}\right\rangle d^3x = \frac{E_0^2}{2\pi}\langle\cos^2\omega t\rangle = \boxed{\frac{E_0^2}{4\pi}}$$

Physics 511

[illegible] #7 Solutions

Pr. 4 Chromatic dispersion in conductors

(a) The Drude model treats a conductor a gas of "free electrons" ~~which~~ whose dynamics are governed by the external field and phenomenological damping.

The current density $\vec{J} = -eN_e\vec{v}$, where N_e is the density of free electrons.

Newton's equation $m_e \frac{d\vec{v}}{dt} = -m\nu_e \vec{v} - e\vec{E}$

ν_e: collision rate

$$\Rightarrow \frac{d\vec{J}}{dt} = -eN_e\frac{d\vec{v}}{dt} = eN_e\nu_e\vec{v} + \frac{N_e e^2}{m_e}\vec{E}$$

$$\Rightarrow \boxed{\frac{d\vec{J}}{dt} + \nu_e\vec{J} = \frac{\omega_p^2}{4\pi}\vec{E}}$$ Generalized Ohm's Law

where $\omega_p^2 = \frac{4\pi N_e e^2}{m_e}$ (Plasma freq^2)

For ~~Random~~ harmonic time dependence (Fourier transform)

$$-i\omega\,\underset{\sim}{\vec{J}}(\omega) + \nu_e\,\underset{\sim}{\vec{J}}(\omega) = \frac{\omega_p^2}{4\pi}\,\underset{\sim}{\vec{E}}(\omega)$$

$$\Rightarrow \boxed{\underset{\sim}{\vec{J}}(\omega) = \underbrace{\frac{\omega_p^2/4\pi}{\nu_e - i\omega}}_{\equiv\, \tilde{\sigma}(\omega)\text{ : Conductivity}}\,\underset{\sim}{\vec{E}}(\omega)}$$

(b) Typically $\omega_p >> \nu_e$ (e.g. in copper $\nu_e = 4\times10^{13}\,s^{-1}$, $\omega_p \cong 10^{16}\,s^{-1}$)

Three different regemes:

(i) Low freq $\omega << \nu_e$

$$\boxed{\sigma(\omega) \approx \frac{\omega_p^2}{4\pi \nu_e} = \sigma_0}$$ "Static" conductivity

(ii) Intermediate $\nu_e << \omega < \omega_p$,

(iii) High freq. $\omega > \omega_p$

$\Rightarrow$ Neglect collisions: $\sigma(\omega) \approx i\frac{\omega_p^2}{4\pi\omega}$

If we take $\mu, \epsilon \approx 1$ (ignore bound charges), the dispersion relation

$$k^2 = \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega}\right) \cong \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\Rightarrow \boxed{c^2k^2 \cong \omega^2 - \omega_p^2}$$ Plasma dispersion relation

Physically this makes sense. If the "drive" frequency is much smaller than the collision rate then the electrons barely notice the external force during the time between collisions $\Rightarrow$ the force is approximately static. On the other hand, if the drive-frequency is very large, then the electrons don't have time to make collisions during any period of oscillation and can be ignored $\Rightarrow$ plasma.

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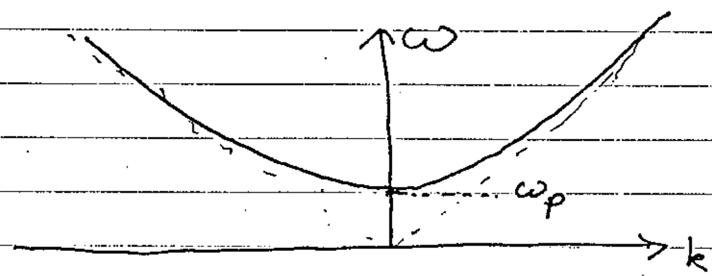
(c) In regimes (ii) and (iii) $\omega \gg \nu_e$

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c} = \left(\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\right)\frac{\omega}{c} = n(\omega)\,\frac{\omega}{c}$$

Frequency dependent index $n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

The dispersion relation $\omega(k) = \sqrt{c^2k^2 + \omega_p^2}$ appears



Note that there are no propagating modes for $\omega < \omega_p$.
Thus, any field below ω_p is said to be below "cutoff".

Explicitly, if $\nu_e \ll \omega < \omega_p$,

$$k = i a \quad \text{where} \quad a = \sqrt{\frac{\omega_p^2}{\omega^2} - 1} \quad \text{is real}$$

$\Rightarrow$ k is <u>pure</u> imaginary $\Rightarrow$ No propagation

since the amplitude of the electric field

$$\underline{E} = E_0 e^{ikz} e^{-i\omega t} = \left(E_0 e^{-az}\right) e^{-i\omega t}$$

Since the wave cannot propagate into the plasma it is all <u>reflected</u>. The small field inside the medium is known as an evanescent wave.

On the other hand if $\omega > \omega_p$, k is purely real $\Rightarrow$ Propagation <u>without</u> attenuation

For wave propagation in a plasma

Phase velocity: $v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > c$

Group velocity: $v_g = \frac{\partial \omega}{\partial k} = \frac{c^2 k}{\sqrt{c^2k^2 + \omega_p^2}} = \frac{c^2 k}{\omega}$

$$\Rightarrow v_g = \frac{c^2}{v_p} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c$$

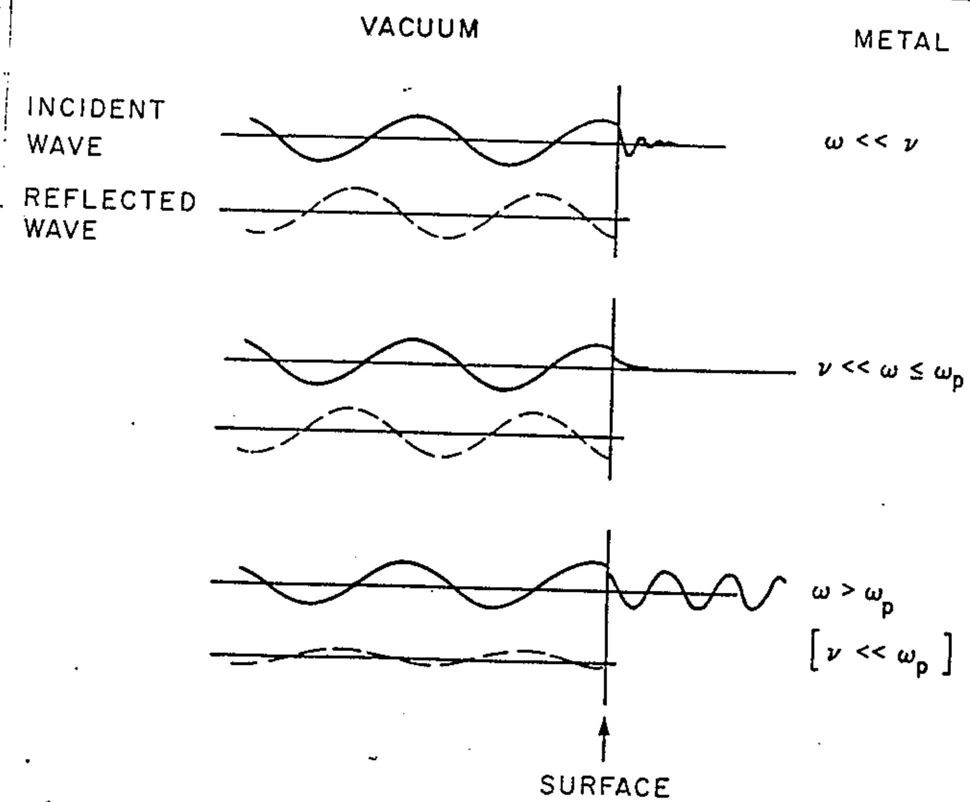


Fig. 6.14 Three possible regimes of waves in a metal. When $\omega \ll \nu$, almost all of the incident energy is reflected. The little that penetrates gets dissipated as heat as a result of electron-lattice ion collisions. When $\nu \ll \omega \leq \omega_p$ all incident energy is reflected; there is no wave in the metal — just exponentially decaying fields. When $\omega > \omega_p$ (but $\nu \ll \omega_p$), the metal is transparent and behaves like a good dielectric. There is a little reflection, as there is from any dielectric. (However, the overall behavior is different in materials like carbon where $\nu \gtrsim \omega_p$.)

From "Electromagnetic Waves & Vibrations"
by G. Bekefi and A. Barrett

(d) Silver: D.C. conductivity $\sigma_0 = 6.17 \times 10^7 / (\text{ohm-m})$
Density (at STP) ~ $10.5\ \text{g/cm}^3$
Valence electrons : 1

Number density $N_e = \dfrac{\rho_m}{m_{Ag}}$ (m_{Ag} = mass of Ag atom)

Atomic weight Ag: 108 amu , 1 amu = mass of proton $= 1.7 \times 10^{-24}\ \text{g}$

$\Rightarrow m_{Ag} = 1.83 \times 10^{-22}\ \text{g}$

$$\Rightarrow \quad N_e = 5.5 \times 10^{22}\ \text{cm}^{-3}$$

D.C. conductivity $\sigma_0 = \dfrac{N_e e^2}{m_e \nu_e} \Rightarrow \nu_e = \dfrac{N_e e^2}{m_e \sigma_0}$

Now for units: Here I will change everything to CGS.

Jackson Table 4: conductivity $\rightarrow$ $\underset{\text{MKS}}{1\ (\text{ohm m})^{-1}} = \underset{\text{CGS}}{9 \times 10^9\ \text{s}^{-1}}$

$\Rightarrow \sigma_0 = 6.17 \times 10^7\ (\text{ohm m})^{-1} = 5.55 \times 10^{17}\ \text{s}^{-1}$
$e = 4.8 \times 10^{-10}\ \text{esu}$

$\Rightarrow$

Collision rate
$$\boxed{\nu_e = \frac{(5.5 \times 10^{22}\ \text{cm}^{-3})(4.8 \times 10^{-10}\ \text{esu})^2}{(9.11 \times 10^{-28}\ \text{g})(5.55 \times 10^{17}\ \text{s}^{-1})} = 2.5 \times 10^{13}\ \text{s}^{-1}}$$

Plasma frequency:
$$\boxed{\omega_p = \sqrt{\frac{4\pi N e^2}{m_e}} = 1.3 \times 10^{16}\ \text{s}^{-1}}$$

Note $\omega_p \gg \nu_e$

(e) The complex index of refraction $n(\omega) = \sqrt{1 + i\,\frac{4\pi\sigma(\omega)}{\omega}}$

• Typical microwave frequency $\quad \omega = 2\pi(5\times 10^{10}\,\text{Hz})$
$\qquad = 3.1\times 10^{11}$

Since $\omega \ll \nu_e$, the conductivity is approximately its static value. Also $\frac{4\pi\sigma_0}{\omega} \gg 1 \Rightarrow$ Excellent conductor

$$\Rightarrow \boxed{n(\omega) \approx \sqrt{i\,\frac{4\pi\sigma_0}{\omega}} \approx \sqrt{\frac{2\pi\sigma_0}{\omega}}\,(1+i) = 3.3\times 10^{3}\,(1+i)}$$

• Typical visible: $\quad \frac{\omega}{2\pi} = 5\times 10^{14}\,\text{Hz} \quad \gg \nu_e$

$\Rightarrow$ Plasma dispersion $\quad \tilde{n}(\omega) \approx \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

Since $\omega < \omega_p$ (regime (iii)) we are below cutoff

$$\Rightarrow \boxed{\tilde{n}(\omega) = i\sqrt{\frac{\omega_p^2}{\omega^2} - 1} \cong 4i}$$

Perfect reflection

• Typical x-ray $\quad \frac{\omega}{2\pi} = 5\times 10^{18}\,\text{Hz} \quad \Rightarrow \quad \omega > \omega_p$

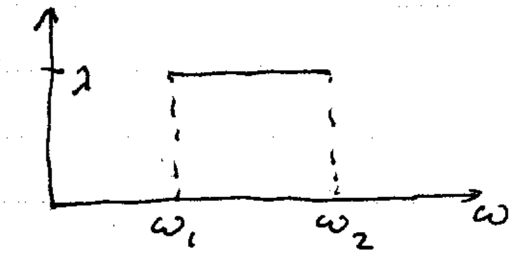
$\Rightarrow$ Plasma above cutoff

$$\boxed{n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \approx 1 - \frac{\omega_p^2}{2\omega^2} = 1 - 8.8\times 10^{-8}}$$

$\Rightarrow$ Silver is essentially transparent

<u>Pr 3</u> Jackson 7.14: Kramers- Krönig

(a) Given: $\epsilon''(\omega)$



$$\epsilon'(\omega) = 1 + 4\pi\chi'(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\Omega\, \epsilon''(\Omega)}{\Omega^2 - \omega^2}\, d\Omega$$

$$= 1 + \frac{\lambda}{\pi} P \int_{\omega_1}^{\omega_2} d\Omega \left(\frac{1}{\Omega - \omega} + \frac{1}{\Omega + \omega} \right)$$ (Having split up the partial fractions)

<u>Case 1:</u> $\omega \notin [\omega_1, \omega_2] \Rightarrow$ The integrand has no poles and the integral can be performed without principal part

$$\Rightarrow \epsilon'(\omega) = 1 + \frac{\lambda}{\pi} \ln \left\{ \frac{\omega_2 - \omega}{\omega_1 - \omega} \cdot \frac{\omega_2 + \omega}{\omega_1 + \omega} \right\}$$

<u>Case 2:</u> $\omega_1 < \omega < \omega_2$: The integrand has a pole at ω:

$\Rightarrow$ We must carefully take the principale part

$$P \int_{\omega_1}^{\omega_2} \frac{d\Omega}{\Omega - \omega} = P \int_{\omega_1 - \omega}^{\omega_2 - \omega} \frac{dx}{x} = \lim_{\delta \to 0} \left[\int_{\omega_1 - \omega}^{-\delta} + \int_{+\delta}^{\omega_2 - \omega} \right] \frac{dx}{x}$$

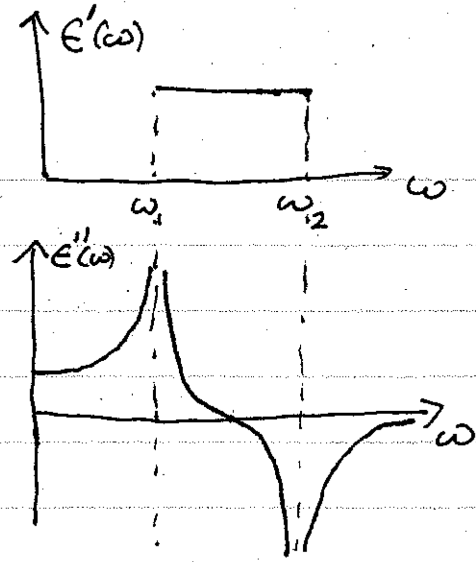
(excluding the pole)

$$= \lim_{\delta \to 0} \log \left\{ \frac{-\delta}{\omega_1 - \omega} \; \frac{\omega_2 - \omega}{\delta} \right\} = \ln \left| \frac{\omega_2 - \omega}{\omega_1 - \omega} \right|$$

$$\int_{\omega_1}^{\omega_2} \frac{d\Omega}{\Omega + \omega} = \ln \left(\frac{\omega_2 + \omega}{\omega_1 + \omega} \right)$$

$$\Rightarrow \boxed{\epsilon'(\omega) = 1 + \frac{\lambda}{\pi} \ln \left| \frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2} \right|}$$ For all ω

Sketch



Imaginary part

Real part

(b) $\epsilon''(\omega) = \dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$, find $\epsilon'(\omega)$

This problem should be familiar to us. By inspection:

$$\frac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} = \mathrm{Im}\left[\frac{\lambda}{\omega_0^2-\omega^2-i\omega\gamma}\right]$$

$\Rightarrow$ We should find $\epsilon'(\omega)-1 = \mathrm{Re}\left[\dfrac{\lambda}{\omega_0^2-\omega^2-i\omega\gamma}\right] = \dfrac{\lambda(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

Now let's use Kramers-Krönig. First, you may worry that $\epsilon''(\omega)$ has poles in the upper-half complex ω-plane. Is this kosher? Yes, remember we require the complex susceptibility to be analytic in upper-half plane. In particular $\mathrm{Im}(\tilde{\chi}(\omega))$ may have poles for $\mathrm{Im}(\omega) > 0$. With that said

$$\epsilon'(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\epsilon''(\Omega)}{\Omega-\omega}\, d\Omega$$

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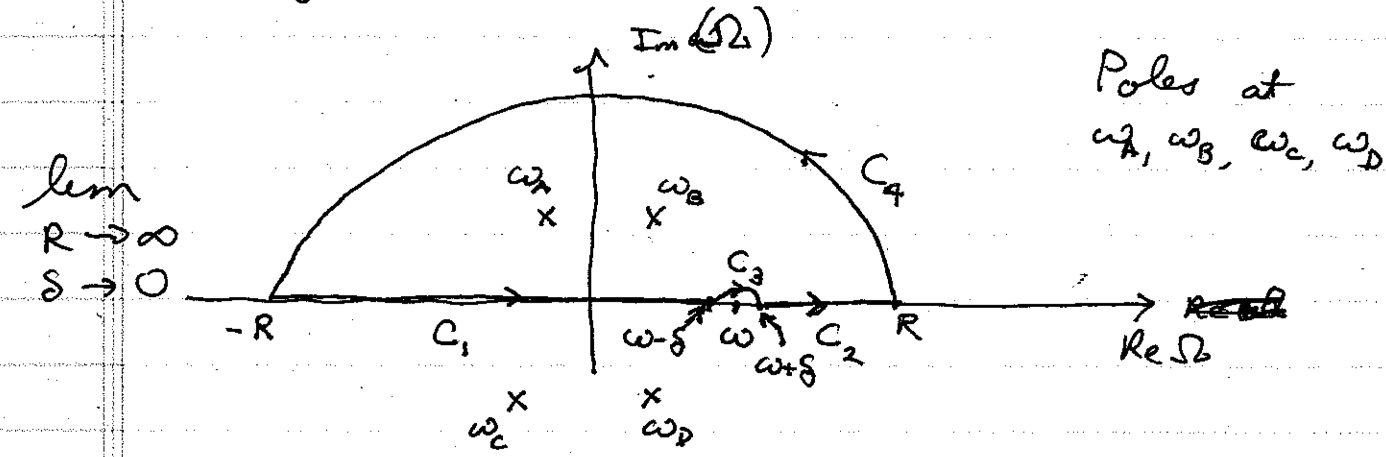
We can do the integration using contour integration in the complex plane. ~~Condens~~ Consider the following contour



$$P\int_{-\infty}^{\infty} \frac{\epsilon''(\Omega)}{\Omega-\omega}\, d\Omega = \left[\int_{C_1} + \int_{C_2}\right] \lim_{\substack{\delta \to 0 \\ R \to \infty}}$$

However $\int_{C_4} \to 0$ in lim $R \to \infty$ because $\epsilon''(\Omega) \to 0$ for $\mathrm{Im}(\Omega) \to \infty$

Thus breaking up the closed contour into its pieces:

$$\oint = \int_{C_1} + \int_{C_3} + \int_{C_2} + \int_{C_4} = 2\pi i \left(\sum \text{Residues}\right)$$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{\epsilon''(\Omega)}{\Omega-\omega}\, d\Omega = 2\pi i \left(\sum \text{Residues}\right) - \lim_{\delta \to 0} \int_{C_3} \frac{\epsilon''(\Omega)\, d\Omega}{\Omega - \omega}$$

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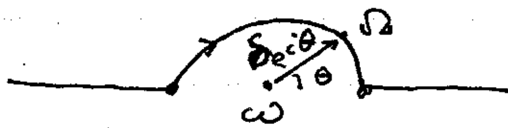
Now: $\lim\limits_{\delta\to 0} \int \frac{\epsilon''(\Omega)}{\Omega-\omega}\, d\Omega = -\pi i\, \epsilon''(\omega)$

Proof:



Using complex phasor

$$\Omega = \omega + \delta e^{i\theta}$$

$$\Rightarrow d\Omega = i\delta e^{i\theta}\, d\theta$$

$$\lim_{\delta\to 0} \int \frac{\epsilon''(\Omega)}{\Omega-\omega}\, d\Omega = \lim_{\delta\to 0} \int_{\pi}^{0} \frac{\epsilon''(\omega+\delta e^{i\theta})}{\delta e^{i\theta}}\, i\delta e^{i\theta} d\theta$$

$$= i\epsilon''(\omega)\int_{\pi}^{0} d\theta = -i\pi\epsilon''(\omega)$$

Finally, the residues:

Poles at $\omega_A = i\frac{\gamma}{2} - \omega'$, $\omega_B = i\frac{\gamma}{2} + \omega'$

$\omega_C = -i\frac{\gamma}{2} - \omega'$, $\omega_D = -i\frac{\gamma}{2} + \omega'$ $\Bigg)$ $\omega' \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$$\text{Residue at } \omega_A = \lim_{\Omega\to\omega_A} \frac{(\Omega-\omega_A)\,\epsilon''(\Omega)}{\Omega-\omega} = \frac{\lambda}{2i\gamma(\omega_A-\omega_B)(\omega_A-\omega}$$

$$\text{Residue at } \omega_B = \lim_{\Omega\to\omega_B} (\Omega-\omega_B)\frac{\epsilon''(\Omega)}{\Omega-\omega} = \frac{+\lambda}{2i\gamma(\omega_B-\omega_A)(\omega_B-\omega}$$

$$\Rightarrow \sum \text{Residues} = \frac{1}{2i}\,\frac{\lambda}{\omega_0^2-\omega^2 - i\gamma\omega}$$

Putting it all together:

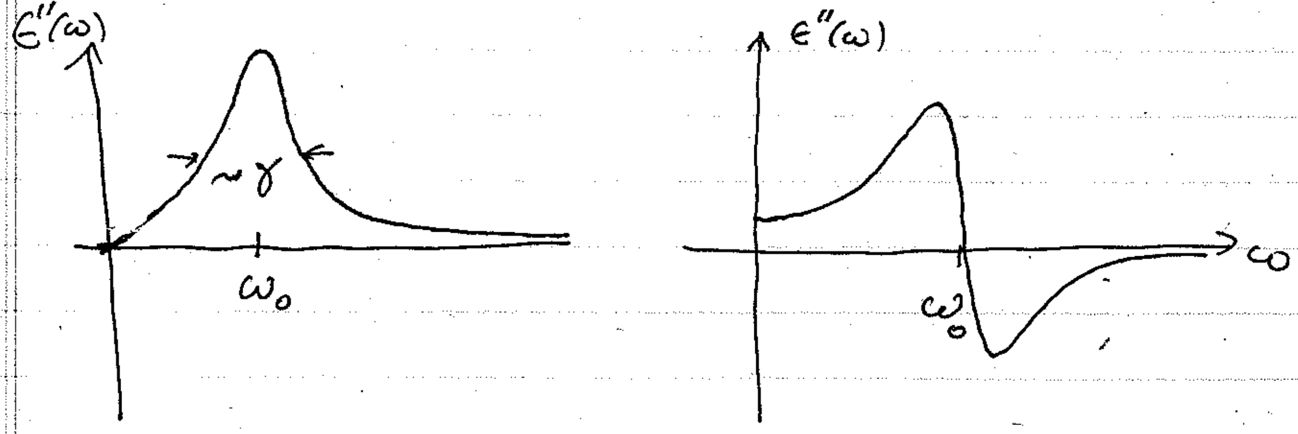
$$\epsilon'(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\epsilon''(\Omega)}{\Omega - \omega}\, d\Omega$$

$$= 1 + 2i \sum \text{Residues} - i\,\epsilon''(\omega)$$

$$= 1 + \frac{\lambda}{\omega_0^2 - \omega^2 - i\omega\gamma} - i \frac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

$$\Rightarrow \boxed{\epsilon''(\omega) = 1 + \frac{\lambda(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}} \quad \text{As expected}$$

Sketch



Usual Lorentz oscillator