Physics 511

Problem Set # 8 Solutions

Pr. 1

Wave propagation in 1D:

Green's function: $\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(z,t) = -4\pi\delta(z)\delta(t)$

$$G(z,t) = 2\pi c\, \Theta(t - |z|/c), \quad \Theta(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$$

Let $\tau = t - |z|/c$ (1D retarded time from $z=0$)

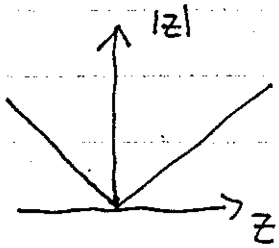
Need: $\frac{d\Theta(x)}{dx} = \delta(x)$: Proof $\left[\int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x)\, dx = -\int_{-\infty}^{\infty} \Theta(x)\frac{df}{dx} \right.$
$= -f(x)\Big|_0^\infty = f(0)$ ✓

$$\Rightarrow \frac{\partial \Theta(\tau)}{\partial t} = \frac{d\Theta}{d\tau}\frac{\partial \tau}{\partial t} = \delta(\tau)$$

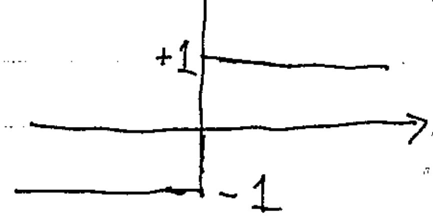
$$\Rightarrow \frac{\partial^2 \Theta(\tau)}{\partial t^2} = \frac{\partial \delta(\tau)}{\partial \tau} \equiv \delta'(\tau) \quad \text{(Derivative of delta)}$$

$$\frac{\partial \Theta(\tau)}{\partial z} = \frac{d\Theta}{d\tau}\frac{\partial \tau}{\partial z} = -\frac{1}{c}\delta(\tau)\frac{d|z|}{dz} = -\frac{1}{c}\delta(\tau)\, \text{sign}(z)$$

Aside:



$\frac{d|z|}{dz} = \text{sign}(z) = 2\Theta(z) - 1$



$$\Rightarrow \frac{\partial^2 \Theta}{\partial z^2}(\tau) = +\frac{1}{c^2}\delta'(\tau) - \frac{1}{c}\delta(\tau)\left(2\delta(z)\right)$$

$$\Rightarrow \Box^{(1)} G(z) = \Box^{(1)} 2\pi c\, \Theta(\tau)$$

$$= 2\pi c \left(\frac{\partial^2 \Theta(\tau)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \Theta}{\partial t^2} \right)$$

$$= 2\pi c \left(\frac{1}{c^2} \delta'(\tau) - \frac{2}{c} \delta(\tau)\, \delta(z) - \frac{1}{c^2} \delta'(\tau) \right)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) 2\pi c\, \Theta\left(t - \frac{|z|}{c}\right) = -4\pi\, \delta\left(t - \frac{|z|}{c}\right) \delta(z)$$

$$= -4\pi\, \delta(t)\, \delta(z)$$

(since $z=0$ only)

q.e.d

(b) Uniform current sheet $\vec{J} = \sigma_0 v_0 \cos\omega t\, \delta(z)\, \hat{x}$

$\rho = \sigma_0\, \delta(z)$

Use the Green's fnct:

$$\Phi(z,t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dz'\, G(z-z', t-t')\, \rho(z-z')$$

$$\Rightarrow \boxed{\Phi(z,t) = 2\pi c\, \sigma_0 \int_{-\infty}^{\infty} \Theta\left(t - t' - \frac{|z|}{c}\right) dt'}$$

$$\vec{A}(z,t) = \frac{1}{c} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dz'\, G(z-z'; t-t')\, \vec{J}(z', t')$$

$$\Rightarrow \boxed{\vec{A}(z,t) = 2\pi v_0\, \sigma_0\, \hat{x} \int_{-\infty}^{\infty} \Theta\left(t - t' - \frac{|z|}{c}\right) dt'}$$

(c) $\vec{E}(z,t) = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$= -2\pi\sigma_0 \int_{-\infty}^{\infty} \left(c\vec{\nabla}\Theta\left(t-t'-\tfrac{|z|}{c}\right) - \frac{v_0}{c}\hat{x}\overset{\cos\omega t'}{\downarrow}\frac{\partial \Theta}{\partial t}\left(t-t'-\tfrac{|z|}{c}\right)\right)$$

Aside: $\vec{\nabla}\Theta(\tau) = \hat{z}\frac{\partial\Theta(\tau)}{\partial z} = \hat{z}\frac{d\Theta}{d\tau}\frac{\partial\tau}{\partial z}$

Here $\tau \equiv t-t'-\frac{|z|}{c}$

$$= -\frac{1}{c}\delta(\tau)\,\text{sign}(z) \quad \text{(see previous page)}$$

$$\frac{\partial\Theta(\tau)}{\partial t} = \frac{d\Theta}{d\tau}\frac{\partial\tau}{\partial t} = \delta(\tau)$$

$\Rightarrow \vec{E}(z,t) = +2\pi\sigma_0\,\text{sign}(z)\hat{z} - 2\pi\frac{\sigma_0 v_0}{c}\int_{-\infty}^{\infty}\delta\left(t-t'-\frac{|z|}{c}\right)\cos\omega t'\,dt'$

$$\Rightarrow \boxed{\vec{E}(z,t) = \underbrace{2\pi\sigma_0\,\text{sign}(z)\,\hat{z}}_{\substack{\text{Longitudinal}\\ \text{static field of}\\ \text{a sheet of charge}}} - \underbrace{\frac{2\pi\sigma_0 v_0}{c}\hat{x}\cos\left(\omega t - \frac{\omega}{c}|z|\right)}_{\substack{\text{Plane wave propagating}\\ \text{in } \pm z \text{ direction,}\\ \text{amplitude } -2\pi\vec{K} \leftarrow \text{surface current}}}}$$

$$\vec{B}(z,t) = 2\pi\sigma_0 v_0 \int \vec{\nabla}\times\left[\Theta(\tau)\,\hat{x}\right]\cos\omega t'\,dt'$$

$$= +2\pi\sigma_0 v_0\,\hat{y}\int\frac{\partial\Theta(\tau)}{\partial z}\cos\omega t'\,dt'$$

$$= -2\pi\frac{\sigma_0 v_0}{c}\,\hat{y}\int\delta(\tau)\cos\omega t'\,dt'\ \text{sign}(z)$$

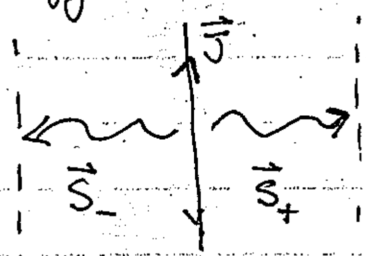
$$\Rightarrow \boxed{\vec{B} = -\frac{2\pi\sigma_0 v_0}{c}\,\hat{y}\,\text{sign}(z)\,\cos\left(\omega t - \frac{\omega}{c}|z|\right)}$$

(d) When we move the charge sheet, an electric field is set up opposite to the direction of the current. Thus we create energy at the rate/area

$$\mathcal{R} = \left\langle \int (-\vec{J}\cdot\vec{E})\,dz \right\rangle = -\left\langle \int_{-\infty}^{\infty} dz\,(\sigma_0 v_0\,\delta(z)\cos\omega t)\left(\frac{-2\pi\sigma_0 v_0}{c}\cos\left[\omega\left(t-\frac{|z|}{c}\right)\right]\right)\right\rangle$$

$$\Rightarrow \boxed{\mathcal{R} = \frac{2\pi\sigma_0^2}{c}v_0^2\,\langle\cos^2\omega t\rangle = \frac{\pi\sigma_0^2 v_0^2}{c}}$$

The energy is radiated in $\pm z$ directions



$$\vec{S}_+ = \frac{c}{4\pi}\vec{E}_+\times\vec{B}_+ = \hat{z}\,\frac{\pi\sigma_0^2 v_0^2}{c}\cos^2(\omega t-kz)$$

$$\vec{S}_- = \frac{c}{4\pi}\vec{E}_-\times\vec{B}_- = \frac{\pi\sigma_0^2 v_0^2}{c}\cos^2(\omega t+kz)\,(-\hat{z})$$

$\Rightarrow$ Total energy flux density $= \langle |\vec{S}_+| + |\vec{S}_-| \rangle$

$$\Rightarrow \quad S_{total} = \frac{2\pi\sigma_0^2 v_0^2}{c}\langle\cos^2(\omega t-kz)+\cos^2(\omega t+kz)\rangle$$

$$\boxed{S_{total} = \frac{\pi\sigma_0^2 v_0^2}{c}}$$

Thus, the energy that is created by moving charges against the field flows outward as electromagnetic energy as is seen at the retarded time $t' = t - \frac{|z|}{c}$

(c) In problem Set #7, we found that the current generated at the surface of a perfect reflector is

$$\vec{J}(z) = \frac{c}{2\pi} E_0 \cos\omega t \, \delta(z) \, \hat{x}$$

The vector potential describing the radiated field is

$$\vec{A} = \frac{1}{c}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dz' \; G(z-z'; t-t') \, \vec{J}(z', t')$$

$$\vec{A} = c\int_{-\infty}^{\infty} dt' \; \Theta(t-t'-|z|) \; E_0 \cos\omega t'$$

(the scalar potential does not contribute to radiation here

⇒ The radiated field is

$$\vec{E}_{rad} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\int_{-\infty}^{\infty} dt' \left[\frac{\partial}{\partial t}\Theta(t-t'-|z|)\right] E_0 \cos\omega t' \, \hat{x}$$

$$= -\int_{-\infty}^{\infty} dt' \; \delta\left(t' - \left(t - \frac{|z|}{c}\right)\right) E_0 \cos\omega t' \, \hat{x}$$

$$= -E_0 \cos\left(\omega t - \frac{\omega}{c}|z|\right)\hat{x}$$

The total field is $\vec{E} = \vec{E}_{inc} + \vec{E}_{rad}$

⇒ $z > 0$ $\quad \vec{E} = 0$ (radiated field destructively interferes with incident wave)

$z < 0$ $\quad \vec{E} = -E_0 \cos\left(\omega t - \frac{\omega}{c} z\right)\hat{x}$

(radiated field generates reflected wave)

Problem 2: Negative Index Materials

Left handed material defined by

$\epsilon(\omega) < 0 \qquad \mu(\omega) < 0$ in some transparent frequency band

(a) Maxwell's equation in isotropic material

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = +i\frac{\omega}{c}\mu\vec{H} \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} - i\frac{\omega}{c}\epsilon\vec{E}$$

Plane wave solution: $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

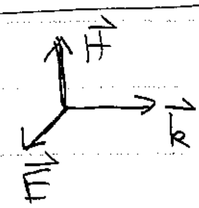
$\vec{H} = \vec{H}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\Rightarrow \vec{k}\cdot\vec{E} = 0 \qquad \vec{k}\cdot\vec{H} = 0$$

$$\Rightarrow \vec{k}\perp\vec{E}, \quad \vec{k}\perp\vec{H}$$

$$i\vec{k}\times\vec{E} = i\frac{\omega}{c}\mu\vec{H} \Rightarrow \frac{c\vec{k}}{\omega}\times\vec{E} = \mu\vec{H}$$

$\Rightarrow$ If $\mu < 0$ $\quad \vec{k}\times\vec{E}$ is in $-\vec{H}$ direction

$\Rightarrow$ 

$(\vec{E}, \vec{H}, \vec{k})$ left-handed coordinate system

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The direction of propagation of energy flow is determined by the Poynting vector

$$\vec{S} \equiv c\frac{\vec{E}\times\vec{H}}{4\pi} \Rightarrow \boxed{\vec{S} \text{ antiparallel to } \vec{k}}$$

$\Rightarrow$ Direction of propagation of plane wave opposite to direction of energy flow

(b) Causality requires, in a transparent band

$$\frac{d}{d\omega}\tilde{\chi}' > -\frac{2}{\omega}\tilde{\chi}'(\omega)$$

where $\tilde{\chi}' \Rightarrow \tilde{\chi}'_e = \frac{\epsilon-1}{4\pi}$ / $\tilde{\chi}' \Rightarrow \tilde{\chi}'_m = \frac{\mu-1}{4\pi}$

for electric/magnetic response and the imaginary parts zero in this band

For $\epsilon<0$ and $\mu<0$ $\quad \tilde{\chi}' < 0$

$$\Rightarrow \boxed{\frac{d}{d\omega}\tilde{\chi}' > \frac{2}{\omega}|\tilde{\chi}'(\omega)|}$$

which is a fine constraint; nothing forbids it

$\Rightarrow$ Must be dispersive

We seek to understand the nature of plane waves propagating in a LH material by studying how radiation is generated by an oscillating plane of current

$$\vec{J}(\vec{x},t) = \vec{j}_0 \, \delta(x - x_0) \, e^{-i\omega t}$$

Ansatz $\quad \vec{E}(\vec{x},t) = \left(\underset{\sim}{\vec{E}}(x)\, e^{-i\omega t}\right) \qquad \vec{\nabla}\cdot\underset{\sim}{\vec{E}} = 0$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\nabla^2 \underset{\sim}{\vec{E}}(x) = -\frac{d^2}{dx^2}\underset{\sim}{\vec{E}}(x)$$

$$= i\frac{\omega}{c}\mu(\vec{\nabla}\times\vec{H}) = i\frac{4\pi}{c^2}\omega\mu\underset{\sim}{\vec{J}}(x) + \frac{\omega^2}{c^2}\mu\epsilon\underset{\sim}{\vec{E}}(x)$$

$$\Rightarrow \boxed{\left(\frac{d^2}{dx^2} + \frac{\omega^2}{c^2}\mu\epsilon\right)\tilde{\vec{E}}(x) = -i\frac{4\pi}{c}\omega\mu j_0 \, \delta(x - x_0)}$$

This is exactly the case solved in problem 1 but with $k_0 \to n\frac{\omega}{c}$ where $k_0 = \frac{\omega}{c}$

$$\Rightarrow \left(\frac{d^2}{dx^2} + k^2\right)\tilde{E}(x) = -i4\pi k\left(\frac{\mu}{n} j_0\right)\delta(x - x_0)$$

$$= -4\pi k(Z j_0)\,\delta(x - x_0)$$

$\Rightarrow$ In medium $j_0 = \sigma v_0 \rightarrow Z j_0$

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⇒ Solution

$$\boxed{E(x,t) = -\frac{2\pi Z}{c} j_0 e^{i n\frac{\omega}{c}|x-x_0|} e^{-i\omega t}}$$

(d) Rate of work done by charges on field

$$\langle \text{Power} \rangle = -\int \langle \vec{J}\cdot\vec{E} \rangle dx = -\frac{\mathrm{Re}}{2}\int \vec{J}^* \cdot \vec{E}\, dx$$

$$= \pi \frac{\mu}{cn} j_0^2$$

Work done per cycle of osica

$$\text{Power} = \frac{dW}{dt} \Rightarrow \left\langle \frac{dW}{dt} \right\rangle = \omega \langle W \rangle$$
$$= \langle \text{Power} \rangle$$

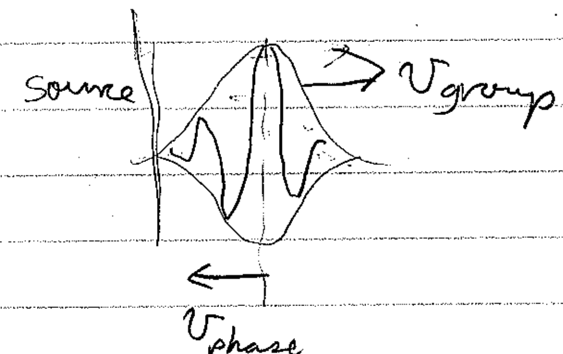
$$\Rightarrow \langle W \rangle = \pi \frac{\mu}{c\omega n} j_0^2 > 0$$

$$\Rightarrow \boxed{\text{If } \mu < 0 \quad n < 0}$$

⇒ Plane waves propagate from ∞ to sources.

⇒ Negative index of refraction

Aside: How does this not violate causality? the waves seem to propagate from the future and effect the present??
The answer is that one never violates causality with a plane wave. They exist everywhere and for all times. A physical wave is a localized packet. The group velocity can be positive while phase velocity negative



(e) The general solution follows from part (c) using the superposition of oscillating sources over all space

General source $J(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{j}_0(x,\omega)\, e^{-i\omega t}$

$$J(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} dx'\, \tilde{j}_0(x',\omega)\, \delta(x-x')$$

Given surface current $\tilde{j}_0(x',\omega)\,\delta(x-x')$ @ x'

$$\Rightarrow \underset{\sim}{E}_x(x,\omega) = -\frac{2\pi}{c}\frac{\mu(\omega)}{n(\omega)}\, \tilde{j}_0(\omega)\, e^{i\frac{\omega}{c} n(\omega)|x-x'|}$$

$\Rightarrow$ Take superposition over harmonic solutions

$$E(x,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} \underset{\sim}{E}_x(x,\omega)\, dx'$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} dx' \left[\frac{-2\pi}{c} \frac{\mu(\omega)}{n(\omega)} \tilde{J}_0(x',\omega)\, e^{i\frac{\omega}{c} n(\omega)|x-x'|} \right]$$

Substitute inverse Fourier transform:

$$\tilde{J}_0(x',\omega) = \int_{-\infty}^{\infty} J(x',t')\, e^{+i\omega t'}\, dt'$$

$$\Rightarrow E(x,t) = -\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \left[\int_{-\infty}^{\infty} d\omega \frac{\mu(\omega)}{n(\omega)} e^{i\frac{\omega}{c}[n(\omega)|x-x'| - c(t-t')]} \right] \frac{J(x',t')}{c}$$

$$\Rightarrow \boxed{E(x,t) = -\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'\, Z(x-x', t-t')\, \frac{J(x',t')}{c} \\ \text{where } Z(x-x',t-t') = \int_{-\infty}^{\infty} d\omega \frac{\mu(\omega)}{n(\omega)} e^{i\frac{\omega}{c}[n(\omega)|x-x'| - c(t-t')]}}$$

(f) If the material has no dispersion

$$Z(x-x';t-t') = \frac{\mu}{n}\int_{-\infty}^{\infty} d\omega \, e^{i\frac{\omega}{c}(n|x-x'| - c(t-t'))}$$

$$= 2\pi \frac{\mu}{n} \, \delta\left(\frac{n}{c}|x-x'| - (t-t')\right)$$

$$= 2\pi c \frac{\mu}{n} \, \delta(n|x-x'| - c(t-t'))$$

Causality requires this to be non zero only for $t > t'$. But this requires $n > 0$

$\Longrightarrow$ No LH material with no dispersion

Epilogue: this problem was taken from "Negative Refractive Index in Left-Handed Material", David Smith and Normal Kroll, Phys. Rev. Lett. 85, 2933 (2000).