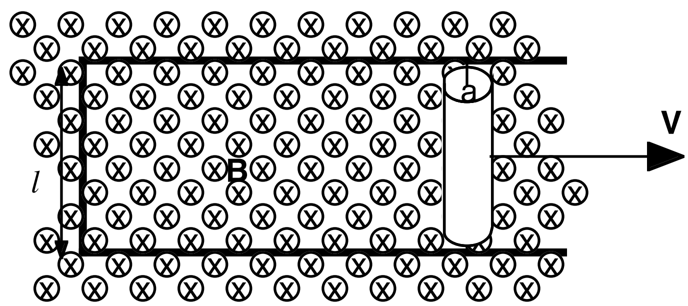
Physics 511

Problem Set #5: Faraday's Law, Quasistatics, and Conservation Laws
DUE Fri. Mar. 10

Read Jackson 3rd Edition, 5.15-5.17, 6.1, 6.7
Low, 2.5-2.6, 3.1-3.3

Problem 1: (10 Points)

A cylindrical resistive material with mass density ρ_m (mass/volume) and conductivity σ slides frictionlessly on two parallel conducting rails. The cylinder has length l and cross sectional area A.. A uniform magnetic field **B**, pointing into the page fills the entire region. The bar moves to the right, starting with a velocity v_0.



(a) What current flows through the circuit, and in what direction?

(b) What is the magnetic force on the rod? In what direction?

(c) Use Newton's equation to show that velocity of the rod as function of time is

$$v(t) = v_0\, e^{-\Gamma t}, \text{ where } \Gamma = \frac{B^2 \sigma}{\rho_M c^2}$$

(d) The initial kinetic energy was $\frac{1}{2} m v_0^2$. At t=∞ the rod looses all this energy. Where does it go? Prove that energy is conserved by showing that the total energy that goes into this sink is $\frac{1}{2} m v_0^2$.

Problem 2: Energy conservation and magnetic fields (15 points)

To see explicitly why we put energy into creating a magnetic field, consider the work per unit length necessary to spin up a long cylindrical shell of charge so that it rotates about its axis at constant angular frequency ω_0. Let the radius of the cylinder be R, the surface charge density be σ, so that $\lambda=2\pi R\sigma$ is the total charge per unit length. After we spin the cylinder up, we have

$$\mathbf{E}_0 = \begin{cases} 0, & \rho < R \\ \dfrac{2\lambda}{\rho}\hat{\mathbf{e}}_\rho, & \rho > R \end{cases}, \qquad \mathbf{B}_0 = \begin{cases} \dfrac{4\pi}{c}\sigma\omega_0 R\hat{\mathbf{e}}_z, & \rho < R \\ 0, & \rho > R \end{cases}.$$

Now suppose we spin the cylinder up from rest, slowly over some time interval T. If T is much greater than R/c, we can neglect the displacement current. Then **B** as a function of time is just the expression above with ω_0 replaced by $\omega(t)$. With $\mathbf{B}(\mathbf{x},t)$ in hand, we can find the electric field as a function of time from Faraday's law:

(a) What is $\mathbf{E}(\mathbf{x},t)$ as a function of time for all space? You may assume that the time dependent part of **E** is in the $\hat{\mathbf{e}}_\phi$ direction.

(b) If you are the external agent spinning up the shell, you must provide energy per unit volume at a rate $-\mathbf{J}\cdot\mathbf{E}$, over and above the energy you must put in to increase the mechanical rotational energy. Show that the volume integral of this expression per unit length along the z-axis is equal to the time rate of change of the total magnetic energy per unit length.

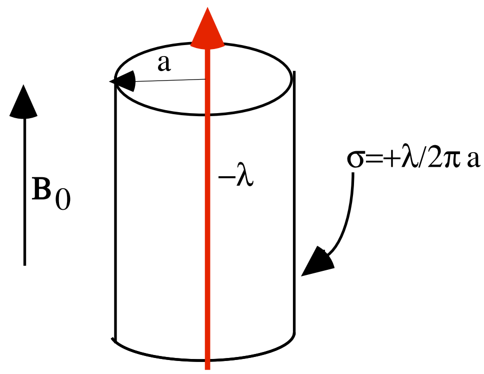
(c) From the above, it is clear that the "source" function for electromagnetic energy, $-\mathbf{J}\cdot\mathbf{E}$, is localized at $\rho = R$. Electromagnetic energy then flows from its creation point into $\rho < R$. Calculate the flow of energy per unit length to $\rho < R$ by integrating the Poynting vector over a cylindrical surface just inside $\rho = R$. Show that the expression is equal to the time rate of change of the total magnetic energy per unit length inside $\rho = R$.

(d) You must also provide an additional torque per unit volume of $-\mathbf{x}\times(\rho_{charge}\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B})$ to spin up the cylinder. Calculate the volume integral of this expression per unit length along the z-axis. Show that the rate of creation of electromagnetic angular momentum at $\rho = R$ is just the energy creation rate divided by ω. Comment on this result.

(e) Where doe the electromagnetic angular momentum you are creating at $\rho = R$ go to? In your answer consider the flux density of electromagnetic angular momentum, $-\mathbf{x}\times(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})$, just inside and outside of $\rho = R$. Where is this electromagnetic angular momentum stored? Can you get it back if you despin the cylinder?

Problem 3: Angular momentum conservation and magnetic fields (15 points)

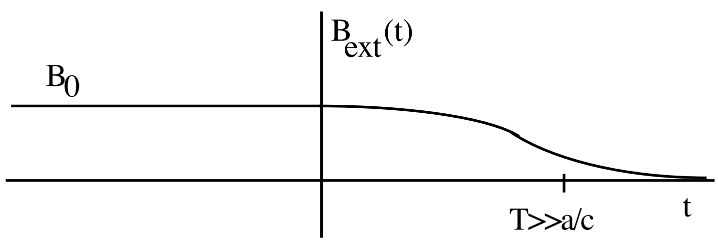
An infinitely long wire carries a charge per unit length of $-\lambda$, and lies along the z-axis. A plastic cylindrical shell of radius a is concentric about the z-axis and has a surface charge density $\sigma = +\lambda / 2\pi a$ uniformly distributed over its surface (see sketch below).



The cylindrical shell is suspended in such a manner that it rotates freely about the z-axis. Its moment of inertia *per unit length* for rotation about its axis is I. The cylinder is initially at rest, and immersed in a uniform external magnetic field $\mathbf{B}_0$, produced by external currents.

(a) Initially, what is the total electromagnetic angular momentum per unit length?

(b) Now assume at $t=0$ we slowly begin to reduce the external currents and B-field to zero over some time $T >> a / c$, $\mathbf{B}_{external}(t) = B(t)\,\hat{\mathbf{e}}_z$



The cylinder will begin to rotate about its axis with angular frequency $\omega(t)$. What is the total magnetic field for $\rho<a$?

(c) From mechanics we know that the time rate of change of angular momentum equals the applied torque (per unit length) $\frac{d}{dt}(I\omega) = (\mathbf{x} \times \lambda \mathbf{E})_z$. Use this relation and Faraday's law to find an expression for $d\omega / dt$

(d) What is the final value of ω (for t>>T)? What is the final value of the magnetic field for $\rho<a$. Show that the final angular momentum of the system (mechanical plus electro-magnetic) for t>T is the same as your part (a) for t<0.