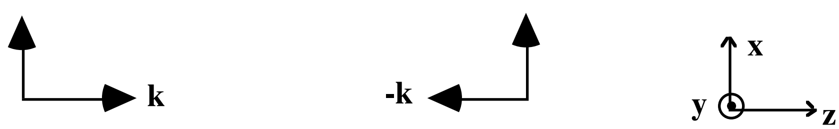
Physics 511

Problem Set #6: Wave Propagation in Free Space
DUE Friday March 31
Read Jackson Chap. 7, Low 3.5-3.7

Required: Problem 1-2 plus your choice of ONE other (Problem 3 OR 4)
(Extra credit for all four problems)

(1) Standing Waves (10 points)

Consider the superposition of two counterpropagating and plane waves with the same frequency, whose electric fields have the same polarization

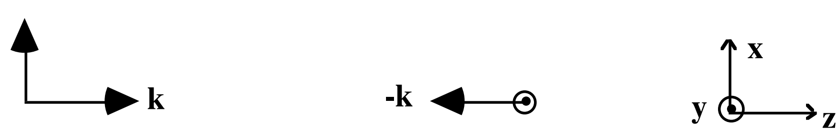


$\mathbf{E}_1 = \hat{\mathbf{x}}\, E_0 \cos(kz - \omega t)$ $\qquad$ $\mathbf{E}_2 = \hat{\mathbf{x}}\, E_0 \cos(kz + \omega t)$

(a) What is the total electric field $\mathbf{E}_3 = \mathbf{E}_1 + \mathbf{E}_2$? What is the total magnetic field, $\mathbf{B}_3$? Sketch $|\mathbf{E}_3(z,t)|$ and $|\mathbf{B}_3(z,t)|$ as a function of z over one wavelength for $t = 0,\ T/4,\ T/2,\ 3T/4,\ T$, where T is the period of oscillation.

(b) What are the electric and magnetic energy densities as a function of z and t., and time averaged? What is the time average energy flux (intensity) - Explain you answer.

Now consider now two the superposition to two counterpropagating traveling planes waves that are cross-polarized and 90∞ out of phase:



$\mathbf{E}_1 = \hat{\mathbf{x}}\, E_0 \cos(kz - \omega t)$ $\qquad$ $\mathbf{E}_2 = \hat{\mathbf{y}}\, E_0 \sin(kz + \omega t)$

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(c) The polarization of this wave is *not* inform in space. Show that, relative to the $+z$-direction the polarization varies over one half wavelength from positive-helicity circular ($z=0$), to linear ($z=\lambda/8$), to negative-helicity circular ($z=\lambda/4$), to linear ($z=3\lambda/8$), and back to positive-helicity circular ($z=\lambda/2$).

(d) Show that the total field can be written as a superposition of a positive-helicity circular, and a negative-helicity circular polarized *standing wave*. You should find that the nodes of one standing wave corresponds to the anti nodes of the other.

(e) What is the intensity as a function of position?

Note: This field configuration is very important in the study of "laser cooling", whereby laser light can be used to cool a gas of atoms.

Problem 2: Spherical Waves (15 points)

Consider the wave equation in three dimensions for a *scalar* field $\psi(\mathbf{r},t)$.

$$\nabla^2\psi - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0\,.$$

We seek solutions for monochromatic wave in spherical coordinates, independent of θ and ϕ - this corresponds, e.g., to waves generated by a point source.

(a) Let $\psi(\mathbf{r},t) = \tilde{\psi}(r)e^{-i\omega t}$. Show that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\psi}}{\partial r}\right) + k^2\tilde{\psi} = 0$, where $k = \omega/\mathrm{v}$.

Show that the most general solution can be written as superpositions of

$$\psi_1(\mathbf{r},t) = u_0\frac{\cos(kr-\omega t-\phi)}{r} \quad \text{and} \quad \psi_2(\mathbf{r},t) = u_0\frac{\cos(kr+\omega t-\phi)}{r}.$$

What is the physical difference between ψ_1 and ψ_2 ?

(b) In free space we know that Maxwell's Equations imply that the waves are transverse. A first guess at the vector spherical wave would choose the polarization in the $\hat{\phi}$ of $\hat{\theta}$ direction. Show that $\mathbf{E} = E_0\frac{\cos(kr-\omega t)}{kr}\hat{\phi}$ does not satisfy Maxwell's Equations.

(c) The simplest possible *vector* spherical wave for the electric field in free space is

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$$\mathbf{E}(r,\theta,\phi) = E_0\left(\frac{\sin\theta}{kr}\right)\left[\cos(kr-\omega t) - \left(\frac{1}{kr}\right)\sin(kr-\omega t)\right]\hat{\phi}.$$

(This corresponds to "magnetic dipole radiation" as we will see)

Show that **E** obeys all four Maxwell's equation, in vacuum, and find the associated magnetic field.

(d) Show that in the limit $kr << 1$, the magnetic field has the *instantaneous* form of a static dipole field $\mathbf{B}(\mathbf{x},t) = \dfrac{3(\mathbf{m}(t)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}(t)}{r^3}$, where $\mathbf{m}(t) = \dfrac{E_0}{k^3}\cos\omega t\,\hat{\mathbf{e}}_z$.

(e) Find the time average Poynting vector; does it point in the expected directions and have the expected fall off with *r*.

(f) Find the flux of energy through a sphere, radius R, centered at the origin, and comment on your result.

Problem 3: The Paraxial Wave Equation (15 Points)

Plane waves propagate in a unique direction, determined by the wave vector **k**. However, these waves are unphysical since they extend over all space and thus contain infinite energy. We know that a laser beam travels nearly unidirectionaly, but has extends only over a finite diameter. The propagation of this "pencil" like beam is described by the "paraxial wave equation", which the subject of this problem.

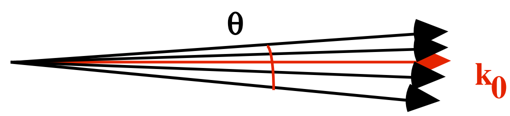
(a) Start with the wave equation $\left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{x},t) = 0$.

Assume a monochromatic wave and an Ansatz $\mathbf{E}(\mathbf{x},t) = \hat{\mathbf{e}}\,\mathcal{E}(\mathbf{x})e^{ik_0 z - \omega t}$ (real part in the end). Here $\mathcal{E}(\mathbf{x})$ is known as the envelope function (for a plane wave, $\mathcal{E}(\mathbf{x})$ would be constant), and $e^{ik_0 z - \omega t}$ as the "carrier wave" which propagates in the z-direction (which will call the longitudinal direction). Under "slowly varying envelope approximation" (SVEA) where the envelope varies of distances large compared to the carrier wave length,

$$\left|\frac{\partial^2\mathcal{E}}{\partial z^2}\right| << k_0\left|\frac{\partial\mathcal{E}}{\partial z}\right|,\ \left|\frac{\partial^2\mathcal{E}}{\partial z^2}\right| << \left|\nabla_T^2\mathcal{E}\right|,\ \text{where } \nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ is the "Transverse Laplacian",}$$

show that the envelope satisfies $i\dfrac{\partial\mathcal{E}}{\partial z} = -\dfrac{1}{2k_0}\nabla_T^2\mathcal{E}$: the paraxial wave equation

(b) Show that the Fourier transform of the field is, $\tilde{\mathbf{E}}(\mathbf{k},\omega) = \tilde{\mathcal{E}}(\mathbf{k} - k_0\mathbf{e}_z)\delta(\omega - c|\mathbf{k}|)$. Let $\mathbf{q} = \mathbf{k} - k_0\mathbf{e}_z$, be the wave vectors of the envelope. Show that the SVEA translates in the Fourier domain, into the "paraxial approximation", i.e., all the wave vectors (rays) make no more than a small angle θ with respect to a main carrier wave vector $k_0\mathbf{e}_z$.



(c) If the characteristic width of the beam in the transverse direction is w_0, use the uncertainty principle to show that $\theta \sim 1/k_0 w_0 = \lambdabar / w_0$ (in optics we call it the "diffraction angle"). Argue qualitatively after a distance on order $z \sim k_0 w_0^2$, the beam width spreads substantially (this is known as the Rayleigh range, or diffraction length).

(d) Notice that the paraxial wave equation has the form of a Schrödinger equation for a free particle in two dimensions, where z plays the role of time, and (x,y) are spatial dimensions. Given the transverse field profile at the input plane z=0, $\mathcal{E}(\mathbf{x}_T,0)$, we can propagate the envelope forward to any other plane as we evolve a wave packet in quantum mechanics:

$$\mathcal{E}(\mathbf{x}_T,z) = e^{iz\nabla_T^2/(2k_0)}\mathcal{E}(\mathbf{x}_T,0) = \int \frac{d^2\mathbf{q}_T}{(2\pi)^2} e^{-i\frac{|\mathbf{q}_T|^2}{2k_0}z}\tilde{\mathcal{E}}(\mathbf{q}_T)e^{i\mathbf{q}\cdot\mathbf{x}},$$

where $\tilde{\mathcal{E}}(\mathbf{q}_T)$ is the Fourier transform of the input beam $\mathcal{E}(\mathbf{x}_T,0)$. Given the initial profile of a Gaussian envelope: $\mathcal{E}(\mathbf{x}_T,0) = E_0 \exp\left(-(x^2+y^2)/w_0^2\right)$, show that

$$\mathcal{E}(\mathbf{x}_T,z) = E_0 e^{-i\phi(z)}\frac{w_0}{w(z)}\exp\left(-(x^2+y^2)/w^2(z)\right)\exp(ik_0(x^2+y^2)/2R(z))$$

where $w(z) = w_0(1+z^2/z_0^2)^{1/2}$, $R(z) = z + z_0^2/z$, $\phi(z) = \tan^{-1}(z/z_0)$,with $z_0 = k_0 w_0^2/2$ (the Rayleigh Range). *Interpret*.

(e) Show that for $z \gg z_0$, the wave fronts are approximately spherical.

(f) There is a fudge in our derivation. We started with the wave equation which assumed transverse waves. Show that our Ansatz with uniform transverse polarization $\hat{\mathbf{e}}$ in fact violates $\nabla\cdot\mathbf{E} = 0$. How can we resolve this dilemma?

Problem 4: Angular momentum in electromagnetic waves (15 points)
(Jackson 2nd Edition: Problem 7.19).

The angular momentum of the electromagnetic field is $\mathbf{L} = \frac{1}{4\pi c}\int d^3x\, \mathbf{x}\times(\mathbf{E}\times\mathbf{B})$, where the integration is over all space.

(a) Eliminate the magnetic field in favor of the vector potential. Show that for field localized to a finite region of space,

$$\mathbf{L} = \frac{1}{4\pi c}\int d^3x\left[\mathbf{E}\times\mathbf{A} + E_i(\mathbf{x}\times\nabla)A_i\right] \quad \text{(sum over components } i \text{ implicit).}$$

The second term is referent to as the "orbital" angular momentum because of the presence of the orbital angular momentum operator familiar in wave mechanics, $\mathbf{L}_{op} = -i(\mathbf{x}\times\nabla)$. The first term relates to the vector nature of the filed itself as is referred to as the "spin" angular momentum.

(b) Consider a Fourier decomposition of the vector potential into transverse, circularly polarized plane waves

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mu=\pm}\int\frac{d^3k}{(2\pi)^3}\left[\tilde{A}_\mu(\mathbf{k})\,\mathbf{e}_\mu(\hat{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + c.c.\right],$$

where $\mathbf{e}_\pm(\hat{\mathbf{k}})$ are circular polarizations orthogonal to $\hat{\mathbf{k}}$.

Show that the *time average* of the "spin" angular momentum is

$$\langle\mathbf{L}_{spin}\rangle = \frac{1}{2\pi c}\int\frac{d^3k}{(2\pi)^3}\,\mathbf{k}\left[\left|\tilde{A}_+(\mathbf{k})\right|^2 - \left|\tilde{A}_-(\mathbf{k})\right|^2\right].$$

(c) Calculate the total energy in the field using the expansion in (b). If we associate, according to quantum mechanics, an energy density of $\frac{\hbar\omega}{V}$ per mode (i.e. Fourier component), what is the "spin" angular momentum of a positive or negative helicity electromagnetic mode, both in direction and magnitude? These are the fundamental characteristics of "photons".