Physics 511 Spring

Problem Set #8: Due Friday April 14, 2006

Problem 1. Electromagnetic radiation in one dimension (10 points)

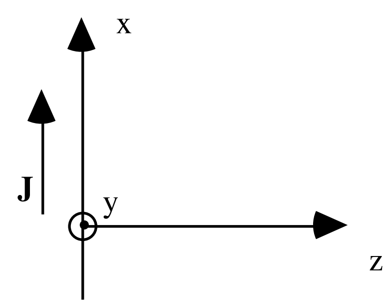
Consider Maxwell's equations with the sources varying only in one spatial dimension (say *z*). In the Lorentz Gauge, the vector a scalar potentials satisfy:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(z,t) = -\frac{4\pi}{c}\mathbf{J}(z,t) \qquad \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(z,t) = -4\pi\rho(z,t)\,.$$

(a) Check that the Green's function, satisfying $\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(z,t) = -4\pi\delta(z)\delta(t)$, is

$G(z,t) = 2\pi c\ \Theta(t - |z|/c)$, where $\Theta(\tau)$ is the Heavyside step function.

(b) Consider a uniform infinite sheet of charge in the *x*-*y* plane, with surface charge density σ_0 , oscillating up and down with velocity $\hat{\mathbf{x}}\, v_0 \cos\omega t$.



Find **A** and Φ, by convolving the sources with the Greens function. Do the integration over *z'* but leave the integration over *t'* to the next step.

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(c) Now calculate **E**(z,t) and **B**(z,t). Be careful to note that $d|z|/dz = \text{sign}(z)$. Show that

$$\mathbf{E}(z,t) = -\frac{2\pi\sigma_0 v_0}{c}\cos[\omega(t-|z|/c)]\hat{\mathbf{x}} + 2\pi\sigma_0 \text{sign}(z)\hat{\mathbf{z}},$$

$$\mathbf{B}(z,t) = -\frac{2\pi\sigma_0 v_0}{c}\cos[\omega(t-|z|/c)]\text{sign}(z)\hat{\mathbf{y}}$$

That is, in addition to the electrostatic field we have electromagnetic waves propagating to the left and right. Show that your solutions satisfy the expected boundary conditions at z=0.

(d) Compare the rate of creation of electromagnetic energy per unit area to the flux density of energy being carried away by the electromagnetic waves.

(Note: If you were shaking this sheet up and down, the electric field of the sheet will oppose you, and you will have to put energy in to keep the sheet moving at a constant amplitude. The electromagnetic force that opposes your effort at z=0 is the *radiation reaction force*. This is sometimes know as the "wave impedance" or "radiation resistance")

(e) In Problem Set #7, problem 2 we considered the reflection of an electromagnetic wave from a perfectly reflecting surface. We can understand this as the waves driving currents on the surface of the conductor which in turn re-radiate waves to the left and right.

Show explicitly that the re-radiated field generates the reflected wave (with the proper amplitude) and exactly destructively interferes with the incident wave.

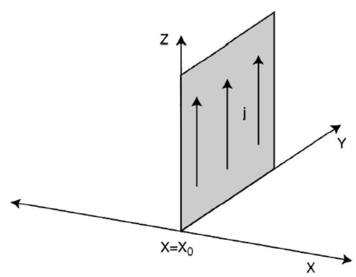
Problem 2: Negative Index of Refraction

In a paper published in 1968 (V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968)), it was predicted that electromagnetic plane waves in a medium having simultaneously negative permittivity and permeability would propagate in a direction opposite to that of the flow of energy.

(a) Use Maxwell's equations to show that a material with $\mu<0$ is "left-handed" (LH) so that **E**, **H**, **k** form a left handed coordinate frame. Show that this implies the direction of propagation of a plane wave front is opposite to the direction of energy flow.

(b) Show that in a frequency band such that a linear material is nearly transparent, having both $\varepsilon(\omega) < 0$ and $\mu(\omega) < 0$ is consistent with causality (i.e. Karmers-Krönig).

Consider a plane of current oscillating with amplitude j_0 in the z-direction at the $x=x_0$ plane. We seek to show that fro $\varepsilon<0$ and $\mu<0$, the waves travel from infinity toward the plane.



(c) Show that the complex amplitude of the electric field satisfies the 1D wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \mu\varepsilon\frac{\omega^2}{c^2}\right)\tilde{E}(x) = -i\frac{4\pi\omega}{c}\mu j_0 \delta(x - x_0)$$

with solution, $E(x,t) = -2\pi\frac{Z}{c} j_0\, e^{i(nk|x-x_0|-\omega t)}$,

where $Z = \frac{\mu}{n}$ is the wave impedance and $n = \pm\sqrt{\mu\varepsilon}$ is the index of refraction.

When $\mu\varepsilon$ is real and positive, there are two mathematically allowed solutions corresponding to n being real-positive or real-negative. The latter is typically thrown out by causality (i.e. keep solution corresponding to retarded time solutions). This is true for media with $\mu>0$. For LH materials we use an alternative criterion.

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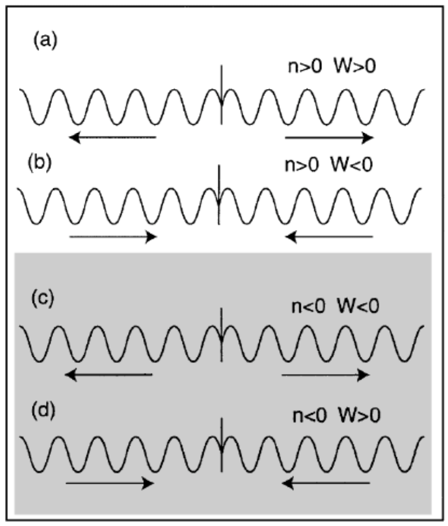
To identify the physical solutions, take the following approach. We require that in radiating waves the time-averaged work done by the sources on the fields be *positive*.

(d) Show that rate is

$$W = \pi \frac{\mu}{cn\omega} j_0^2,$$

and thus show that in RH materials, causal plane wave radiation flows away from the source with $n>0$, while in LH materials, the waves travels toward source with $n<0$.



(e) Show that the general solution to the 1D wave equation has the form

$$E(x,t) = -\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx'\, Z(x-x', t-t') \frac{J(x',t')}{c},$$

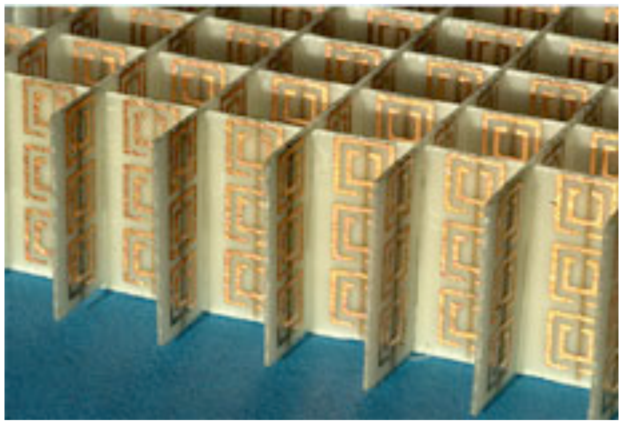
where $Z(x-x', t-t') = \int_{-\infty}^{\infty} d\omega \frac{\mu(\omega)}{n(\omega)} \exp\left[i\frac{\omega}{c}\left(n(\omega)|x-x'| - c(t-t') \right) \right].$

(f) If the material has no dispersion (i.e. μ and n have no frequency response), show

$$Z(x-x', t-t') = 2\pi \frac{\mu}{n} \delta\left(n|x-x_0| - c(t-t') \right),$$

and thus causality requires, $n>0$. Thus there are no left handed materials with $n<0$ without dispersion in the frequency band.

Epilogue: No natural materials occur with $\mu<0$, so this work was seen as purely academic. More recently, however, it was shown by Pendry (J. Pendry *et al.*, Phys. Rev. Lett. **76**, 4773 (1996)) that a *composite* meta-material composed of metal posts interspersed with an array of split-ring resonators produced a finite frequency band with simultaneous *negative permittivity and permeability* for microwaves.



(From Physics Today, June 2004)

The thin wire posts can be described by the dielectric function

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \text{ (a plasma below cutoff),}$$

and the split ring resonator by

$$\mu(\omega) = 1 - \frac{F\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}.$$

The existence of negative phase delay was demonstrated. Such materials have great promise to change the face of optics in that they "reverse light rays". The jury is still out on the ultimate limits imposed by physics.