Physics 511  Electrodynamics

Problem Set #10
Due Friday May 4

Problem 1. Scattering in one dimension (10 points)
Consider a system in which all variation in the material properties vary only in one
direction (call it the \( z \) direction); e.g. a infinite slab of dielectric, homogeneous along the
transverse dimension.

(a) Assuming no free charge density, use the results from problem #1 of P.S. #8 to show that
the formal solution for the electric field is:

\[
E(z,t) = E_{inc}(z,t) - \frac{2\pi}{c} \int dz' J(z', t - |z - z'|/c),
\]

where \( E_{inc} \) is the incident field and \( J \) is the current density. Note that the radiated field is
proportional to the current, and is 180° out of phase with the incident radiated field.

(b) For the case of a monochromatic incident plane wave, \( E_0 e^{i(kz-\omega t)} \), and linear dielectric
consisting of a smooth distribution of scatterers with linear polarizability \( \alpha \) and density \( N(z) \),
show that we arrive at a formal integral equation solution for the complex amplitude,

\[
E(z) = E_0 e^{ikz} + i2\pi\alpha k \int dz' e^{ik(z-z')}N(z')E(z'),
\]

Note for \( \alpha \) real, the scattered wave is 90° out of phase with the incident wave.

(c) Now suppose we have a very thin dielectric slab of thickness \( \Delta z \) and dilute particle
density \( N_0 \). Show that in the first Born approximation, the forward scattered wave is,

\[
E(z) = E_0 e^{ikz} (1 + i(2\pi N_0 \alpha)k\Delta z) \approx E_0 e^{ikz} e^{i(n-1)k\Delta z},
\]

where \( n \) is the index of refraction.
Interpret this important result!

(d) The first Born approximation can give unphysical results if we are not careful in
interpreting them. Consider the case of a nonabsorbing dielectric sheet at the origin, with
surface density \( N \), so that the volume density is \( N(z)=N\delta(z) \). This can be though of as a
beam splitter which partially transmits part of the wave and reflects the rest. Show that under
the first Born approximation, energy is not conserved, i.e. if we define transmission and reflection coefficients as \( t \equiv E_{\text{trans}} / E_0 \), and \( r \equiv E_{\text{ref}} / E_0 \), \(|t|^2 + |r|^2 \neq 1\).

(e) For this special problem we can solve the integral equation \textit{exactly} to all orders. Do it, and show that now energy is conserved. \textit{(Hint:} The exact solution will have the form \( E(0)e^{ikt} \)).

\[ \textbf{Problem 2. Radiation from the hydrogen atom (10 Points) from Jackson} \]

Consider the hydrogen atom in a superposition of the ground state \((1s)\) and the first exited state \((2p, m=0)\), where \(m\) is so-called the “magnetic quantum number”. The effective charge density, associated with the electron motion relative to the nucleus, is given by product wave functions

\[
\rho(r, \theta, t) = -e\psi_{2p,0}^{*}(x,t)\psi_{1s}(x,t) + c.c. = 2\text{Re}\left(-e\psi_{2p,0}^{*}(x,t)\psi_{1s}(x,t)\right)
= \text{Re}\left(\frac{4e}{\sqrt{6}}a_0^{-4}e^{-3r/2a_0}Y_{00}Y_{10}(\theta) e^{-i\omega t}\right),
\]

where \( a_0 = \hbar^2 / me^2 \) is the Bohr radius and \( \hbar\omega_0 = 3e^2 / 8a_0 \) is the energy difference between the two levels.

(a) Evaluate all the radiation multipoles in the long wavelength limit.

(b) In the electric dipole approximation, calculate the total time-average power radiated. Express your answer in units of \( \alpha^4 (\hbar\omega_0)c / a_0 \) where \( \alpha = e^2 / \hbar c \) is the fine-structure constant.

(c) Estimate the transition rate between energy levels as the classical power divided by the photon energy \( \hbar\omega_0 \).

(d) If, instead of the semiclassical charge density used above, we described the \( 2p \) state by a \textit{circular} Bohr orbit of radius \( 2a_0 \), with the electron moving at the angular velocity \( \omega_0 \), what would the power be? Express your answer in the same units as in (b) and evaluate the ratio of the two powers numerically. Please comment on your result.
Problem 3: Scattering by a lossy particle (10 points) from Jackson

An unpolarized plane wave of frequency $\omega$ is scattered by a *slightly lossy* uniform isotropic dielectric sphere of radius $R$ much smaller than the wavelength. The sphere is characterized by an ordinary real dielectric constant $\varepsilon$ and real conductivity $\sigma$. The parameters are such that the skin depth is very large compared to the radius $R$.

(a) Calculate the differential and total *scattering* cross section
(b) Show that the *absorption* cross-section is $\sigma_{abs} = 48\pi^2 R^2 \frac{R\sigma / c}{(\varepsilon + 2)^2 + (4\pi\sigma / \omega)^2}$