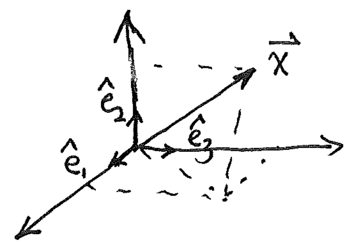
Phys511 Lecture # 2: Vectors and Tensors

Basic Review: Euclidean geometry and tensor notation

3D Cartesian space: position vector $\vec{x}$



$$\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = \sum_{i=1}^{3} x_i \hat{e}_i$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{"Representation in a basis"}$$

Right handed coordinate basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

"Einstein Summation convention": Sum $i = 1 \to \dim$ if index is repeated $\Rightarrow \vec{x} = x_i \hat{e}_i$

Inner product (dot) $\Rightarrow$ Defines length and distance

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 \\ 0 \end{cases} \quad \text{orthonormal vectors}$$

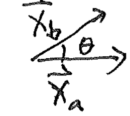
Notation $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (Kroneker delta)

Linear operation $\vec{x}_a \cdot (c_1 \vec{x}_b + c_2 \vec{x}_c) = c_1 (\vec{x}_a \cdot \vec{x}_b) + c_2 (\vec{x}_a \cdot \vec{x}_c)$

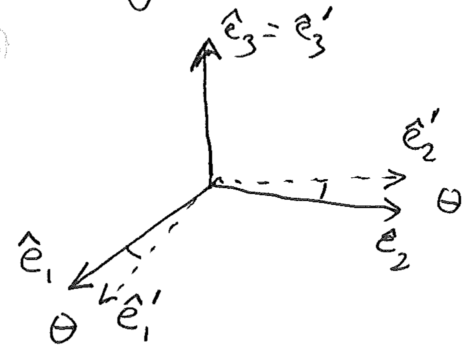
$$\Rightarrow \vec{x} \cdot \vec{x} = \underbrace{(x_i \hat{e}_i) \cdot (x_j \hat{e}_j)}_{\text{Note never reuse dummy indecies}} = x_i x_j (\hat{e}_i \cdot \hat{e}_j) = x_i x_j \delta_{ij} = x_i x_i$$

$$\Rightarrow \vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + x_3^2 = |\vec{x}|^2$$ Length of $\vec{x}$ squared (by Pythagorus)

Law of Cosines $\Rightarrow \vec{x}_a \cdot \vec{x}_b = |\vec{x}_a| |\vec{x}_b| \cos\theta_{ab}$



Change of basis $\Rightarrow$ Definition of vector



example rotation of coordinate system

$$\hat{e}_1' = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$$
$$\hat{e}_2' = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2$$
$$\hat{e}_3' = \hat{e}_3 \qquad \text{(Passive: Rotating basis)}$$

$$\hat{e}_i' = R_{ij}\, \hat{e}_j \qquad \underset{\approx}{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

$$\hat{e}_i' \cdot \hat{e}_j' = (R_{ik}\, \hat{e}_k) \cdot (R_{jl}\, \hat{e}_l) = R_{ik} R_{jl}\, \hat{e}_k \cdot \hat{e}_l = R_{ik} R_{jl}\, \delta_{kl}$$
$$= R_{ik} R_{jk} = (\underset{\approx}{R})_{ik} (\underset{\approx}{R}^T)_{kj} = (\underset{\approx}{R}\, \underset{\approx}{R}^T)_{ij}$$

But $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$ (lengths are preserved in rotation)
$$= (\underset{\approx}{\mathbb{1}})_{ij}$$

$$\Rightarrow \underset{\approx}{R}\, \underset{\approx}{R}^T = \underset{\approx}{\mathbb{1}} \Rightarrow \boxed{\underset{\approx}{R}^{-1} = \underset{\approx}{R}^T}$$

Orthogonal transformation

$\det(\underset{\approx}{R}) = \pm 1$

Two kinds of orthogonal transformations

- "Proper" (<u>rotations</u>)

  $\det(\underset{\approx}{R}) = +1$

- "Improper" (<u>reflections</u>)

  example $\hat{e}_1' = \hat{e}_1$, $\hat{e}_2' = \hat{e}_2$

  $\hat{e}_3' = -\hat{e}_3$

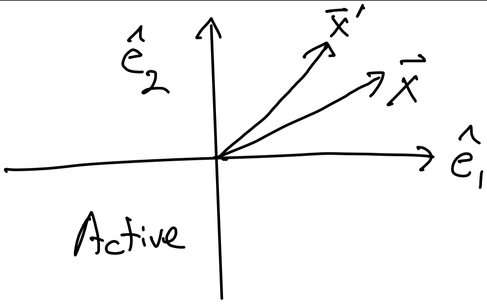
  $\det(\underset{\approx}{R}) = -1$

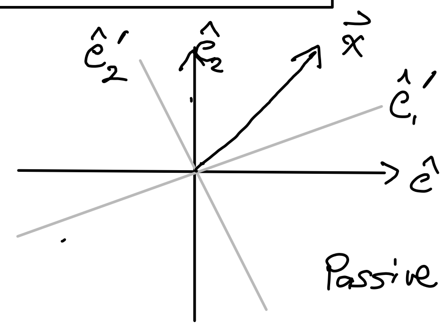
Transformation of components

Active Rotation: $\vec{x}' = \underset{\approx}{R}\,\vec{x} = \underset{\approx}{R}\,(x_i \hat{e}_i) = x_i\, \underset{\approx}{R}\, \hat{e}_i = x_i R_{ij} \hat{e}_j$

$\Rightarrow$ New coordinates with respect to old basis

$$\boxed{x'_j = \hat{e}_j \cdot \vec{x}' = x_i R_{ij} = R^T_{ji} x_i}$$





Passive rotation: $\vec{x} = x_i \hat{e}_i = x'_j \hat{e}'_j$ (Same geometric point)

$\Rightarrow x_i \hat{e}_i = x'_j R_{jk} \hat{e}_k = (R^T_{kj} x'_j)\, \hat{e}_k \Rightarrow \boxed{x'_j = (\underset{\approx}{R}^T)^{-1}_{ji} x_i = R_{ji} x_i}$

Passive

Abstract Definition of a Vector

Geometrical object $\vec{V} = \sum_{i=1}^{3} V_i \hat{e}_i = V_i \hat{e}_i$

Define vector such that components of $\vec{V}$ transform as the components of $\vec{x}$ under orthogonal transformation

$$\boxed{V'_i = R_{ij} V_j}$$

(Here passive rotation, often R_{ij} defined for active)

Example: momentum $\vec{p} = m \frac{d\vec{x}}{dt}$, m, t scalars

Cross-products and pseudo-vectors

Given an orthonormal basis we define:

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\,\hat{e}_k \quad : \quad \underline{\text{Vector cross product}}$$

- Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1 & i \neq j \neq k \text{ (and/or)} \quad i,j,k = \text{cylic } 1,2,3 \\ -1 & \text{same} \quad \text{non cylic} \\ 0 & \text{otherwise} \end{cases}$$

$$\left.\begin{cases} \epsilon_{1,2,3} = \epsilon_{3,1,2} = \epsilon_{2,3,1} = 1 \\ \epsilon_{1,3,2} = \epsilon_{2,1,3} = \epsilon_{3,2,1} = -1 \end{cases}\right\}$$

ϵ_{ijk} is antisymmetric w.r.t exchange of any two indecies

- Important relations

- $$\epsilon_{ijk} U_i V_j W_k = \det\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix}$$

$$\Rightarrow \vec{C} = \vec{A} \times \vec{B} = (\hat{e}_i \times \hat{e}_j) A_i B_j = \epsilon_{ijk}\,\hat{e}_k A_i B_j$$

$$\vec{C} = \epsilon_{ijk} A_j B_k\,\hat{e}_i \quad \Rightarrow \boxed{(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k}$$

Using determinant: $$\vec{A} \times \vec{B} = \det\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$$

- Contracting on any two indecies

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Using tensor notation to prove vector identities

e.g. $\vec{D} \equiv \vec{A} \times (\vec{B} \times \vec{C})$

$$D_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m$$

$$= [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}] A_j B_l C_m = A_j B_i C_j - A_j B_j C_i$$

$$= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B})$$

$$\Longrightarrow \boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})}$$

Is the cross product between to vectors a vector?

Let $\vec{w} = \vec{u} \times \vec{v} \Rightarrow w_i = \epsilon_{ijk} u_j v_k$

In another orthogonal coordinate system: $w_i' = \epsilon_{ijk}$

$$w_i' = \epsilon_{ijk} R_{jl} R_{km} u_l v_m = \epsilon_{njk} \delta_{in} R_{jl} R_{km} u_l v_m$$

Insert $\delta_{in} = R_{ip} R_{np}$

$$\Rightarrow w_i' = R_{ip} \underbrace{\epsilon_{njk} R_{np} R_{jl} R_{km}}_{\det[\underset{\approx}{R}]\, \epsilon_{plm}} u_{lm}$$

$$\Rightarrow w_i' = \det[\underset{\approx}{R}]\, R_{ip} (\vec{u} \times \vec{v})_p = \det[\underset{\approx}{R}]\, R_{ip} w_p$$

$\Longrightarrow$ **$\vec{w}$ is a pseudo vector**

Tensors:

Consider rigid body rotation:

$$\vec{L} = \sum_{\alpha \atop \text{(particles)}} \vec{x}_\alpha \times (m_\alpha \vec{v}_\alpha) = \sum_\alpha m_e \vec{x}_\alpha \times (\vec{\omega} \times \vec{x}_\alpha)$$

$$= \sum_\alpha m_\alpha \left(r_\alpha^2 \vec{\omega} - \vec{x}_\alpha (\vec{x}_\alpha \cdot \vec{\omega})\right) = \overleftrightarrow{I} \cdot \vec{\omega}$$

$$\overleftrightarrow{I} = \sum_\alpha m_\alpha \Big(r_\alpha^2 \underbrace{\mathbb{1}}_{\text{unit tensor}} - \underbrace{\vec{x}_\alpha \vec{x}_\alpha}_{\text{"Outer product"}}\Big)$$

In components: $L_i = I_{ij} \omega_j$

$$I_{ij} = \sum_\alpha m_\alpha \left(r_\alpha^2 \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)}\right)$$

In a new coordinate system $\delta_{ij} \to \delta_{ij}$ $\quad x_i^{(\alpha)} \to x_i'^{(\alpha)}$

$$I'_{ij} = \sum_\alpha m_\alpha \left(r_\alpha^2 \delta_{ij} - R_{ik} x_k^{(\alpha)} R_{jl} x_l^{(\alpha)}\right)$$

$$\Rightarrow \boxed{I'_{ij} = R_{ik} R_{jl} I_{kl}}$$

<u>Definition</u>: A rank-n tensor is defined such that its ~~comp~~ n components transform as

$$\boxed{T'_{i_1 i_2 \cdots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} T_{j_1 j_2 \cdots j_n}}$$

- **Example of a tensor**: The outer product ~~of between~~ (tensor product) between n vectors is a rank-n tensor

$$T_{ijk} = U_i V_j W_k \quad \text{(obvious)}$$

- **Summing (contracting) on any indecies** (tracing over indecies) reduces the rank by two

e.g. $\underset{\text{sum over } i}{T_{iik}} = (\vec{U} \cdot \vec{V}) U_k \Leftarrow$ rank 1 tensor (vector)

$$M_{ii} = M_{11} + M_{22} + M_{33} \quad : \quad \text{Trace operator in matrix algebra}$$
$$= \mathrm{Tr}(\underset{\approx}{M})$$

- **Relation between cross-product and antisymmetric tensor**

Let $T_{ij} = \epsilon_{ijk} t_k \Rightarrow \underset{\approx}{T} = \begin{bmatrix} 0 & t_3 & -t_2 \\ -t_3 & 0 & t_1 \\ t_2 & -t_1 & 0 \end{bmatrix}$

antisymmetric rank-2 tensor $\equiv$ components of a cross product

$$t_k = (\vec{V} \times \vec{W})_k = \epsilon_{klm} V_l W_m$$

$$\Rightarrow T_{ij} = V_i W_j - V_j W_i \quad \text{(Anti-symmetrized outer product)}$$

t_k and T_{ij} are "duals" $\quad \boxed{t_i = \frac{1}{2} \epsilon_{ijk} T_{jk}}$