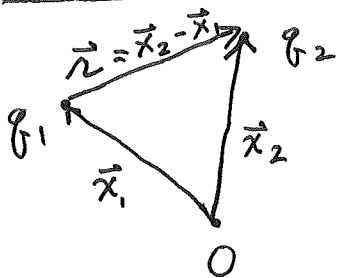


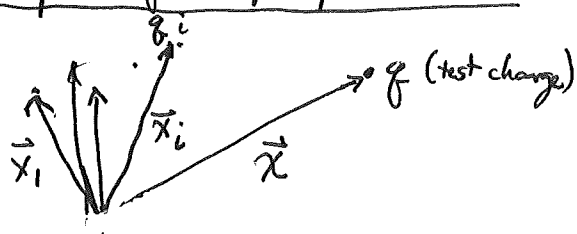
# Physics 511 Lecture # 3 - Review Electrostatics I

## Coulomb's Law



$$\vec{F}_{2,1} = \frac{q_1 q_2}{r^2} \hat{r} = \frac{q_2 q_1}{r^3} \vec{r} = q_1 q_2 \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3}$$

## Principle of superposition



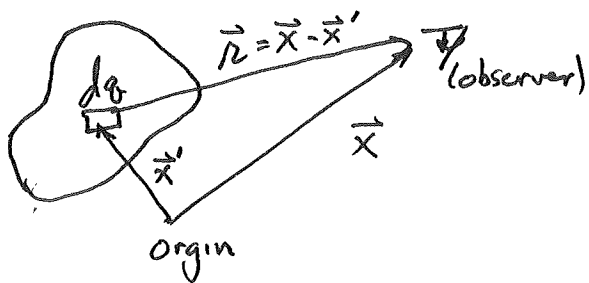
$$\vec{F}(\vec{x}) = \sum_i \frac{q q_i}{|\vec{x} - \vec{x}_i|^3} (\vec{x} - \vec{x}_i)$$

$$= q \sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|^3} (\vec{x} - \vec{x}_i)$$

$\vec{E}(\vec{x})$  (Electric field of charge distribution)

$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|^3} (\vec{x} - \vec{x}_i)$$

continuous  $\int \frac{dq(\vec{x}')}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}')$



Distribution

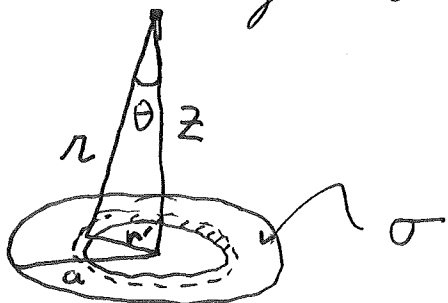
• Volume:  $dq = \rho(\vec{x}') d^3\vec{x}'$

$$d^3\vec{x}' = dx' dy' dz'$$

• Surface:  $dq = \sigma(\vec{x}') da'$

• line  $dq = \lambda(\vec{x}') dl'$

Example: Charged disk, uniform surface density  $\sigma$



$$dq = (2\pi r' dr') \sigma$$

$$R = \sqrt{r'^2 + z^2}$$

Observation point on axis

Use symmetry:  $\vec{E}(z) = \vec{e}_z E_z(z)$  (most point in  $z$ -direction)

L3.2

$$dE_z = \frac{dq}{r^2} \cos\theta = \frac{dq}{r^3} z$$

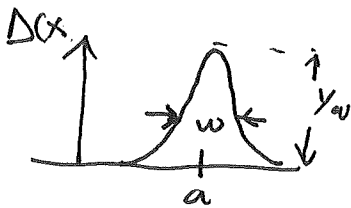
$$\Rightarrow E_z(z) = 2\pi\sigma z \int_0^a \frac{r' dr'}{(r'^2 + z^2)^{3/2}} = 2\pi\sigma z \left[ (r'^2 + z^2)^{-1/2} \right]_0^a$$

$$= 2\pi\sigma z \left[ \frac{1}{z} - \frac{1}{\sqrt{a^2 + z^2}} \right] = 2\pi\sigma \left[ 1 - \frac{1}{(1 + \frac{a^2}{z^2})^{1/2}} \right]$$

$\lim_{\frac{a}{z} \rightarrow 0}$  (infinitely far away)  $(1 + \frac{a^2}{z^2})^{-1/2} \approx 1 - \frac{1}{2} \frac{a^2}{z^2}$   
 (recall  $(1+x)^\delta \approx 1 + \delta x$   $\delta \ll 1$ )

$$\Rightarrow E_z(z) \approx \frac{\pi a^2 \sigma}{z^2} = \frac{Q}{z^2} = \frac{Q}{r^2} \checkmark$$

Dirac Delta function for "point" distributions



$$\delta(x) = \lim_{w \rightarrow 0} \Delta(x)$$

$$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases} \quad \text{"Distribution"}$$

Properties

$$(i) \int_{-\infty}^{\infty} dx \delta(x-a) = 1$$

$$(ii) \int_{-\infty}^{\infty} dx \delta(x-a) f(x) = f(a)$$

$$(iii) \delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|df/dx(x_i)|}$$

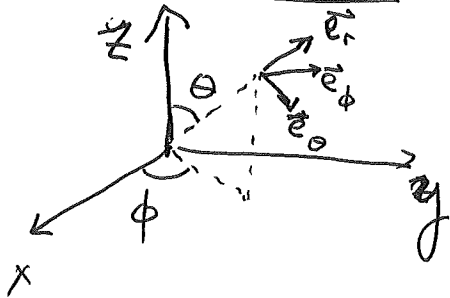
(iv)  $\delta$  has units  $\frac{1}{[x]}$

Higher dimensional delta funct.  $\delta^{(3)}(\vec{x} - \vec{x}') = \delta(x-x') \delta(y-y') \delta(z-z)$ : Cartesian expansion  
 $\therefore$  density of a point  
 $\int d^3x \delta^{(3)}(\vec{x} - \vec{x}_0) = 1$

• Charge density of a point:  $\rho(\vec{x}) = q \delta^{(3)}(\vec{x} - \vec{x}')$   
charge at  $\vec{x}'$

• Charge density of an infinite plane of surface charge  $\sigma$   
(say the  $x$ - $y$  plane at  $z_0$ )  $\rho(\vec{x}) = \sigma \delta(z - z_0)$

Curvilinear coordinate (e.g. spherical)



$$d^3x = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$  : depend on position

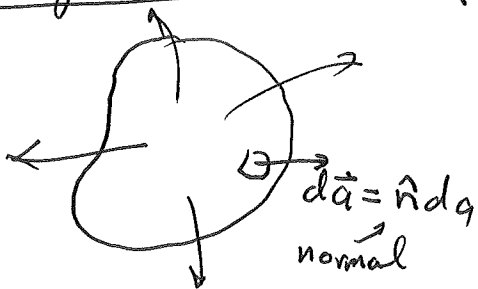
$$\vec{x} = r \vec{e}_r$$

$$\Rightarrow \delta^{(3)}(\vec{x} - \vec{x}') = \frac{\delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi')}{r^2 \sin\theta}$$

## Field equations

Any vector field defined by its divergence + curl  
(plus boundary condition at  $\infty$ )

Divergence: Consider flux of field  $\vec{A}$  through closed surface



$$\oint_S \vec{V} \cdot d\vec{a} = \text{flux of } \vec{V}$$

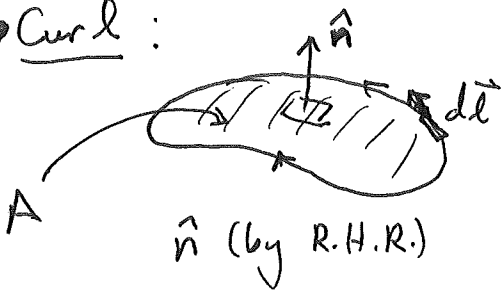
define  $\text{div } \vec{V} = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{V} \cdot d\vec{a}}{\Delta V}$  (Flux into or out of a point)

⇒ Divergence theorem

L3.4

$$\int_V (\text{div } \vec{V}) d^3x = \oint_S \vec{V} \cdot d\vec{a}$$

• Curl:



$$\oint_C \vec{V} \cdot d\vec{l} = \text{Circulation of } \vec{A}$$

$$\text{Define } \vec{n} \cdot \text{curl } \vec{V} = \lim_{\Delta a \rightarrow 0} \frac{\oint_C \vec{V} \cdot d\vec{l}}{\Delta a}$$

$$\Rightarrow \int_S d\vec{a} \cdot (\text{curl } \vec{V}) = \oint_C \vec{V} \cdot d\vec{l}$$

Stoke's theorem

Note general form of integral calculus

$$\int_{\mathcal{O}} DF d\tau^{(N)} = \int_{\partial\mathcal{O}} F d\tau^{(N-1)}$$

integral of DF on object  $\mathcal{O}$       antiderivative on the boundary  $\partial\mathcal{O}$

"Del operator"  $\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = e_i \partial_i$

Can show /  $\text{div } \vec{V} = \vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \partial_i V_i$

$$\left\{ \begin{aligned} \text{curl } \vec{V} &= \vec{\nabla} \times \vec{V} = \vec{e}_i \epsilon_{ijk} \partial_j V_k \end{aligned} \right.$$

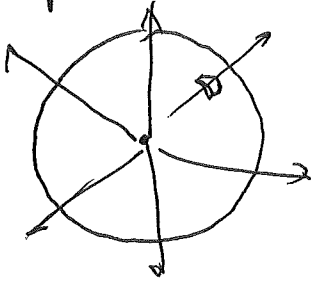
• Gradient: Given scalar field  $\phi$ :  $\text{grad } \phi = \vec{\nabla} \phi$

Integral  $\int_{\vec{x}_a}^{\vec{x}_b} \vec{\nabla} \phi \cdot d\vec{l} = \phi(\vec{x}_b) - \phi(\vec{x}_a)$  (independent of path)

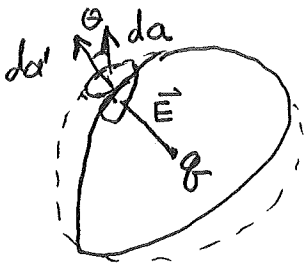
# Field equations for Electrostatic Fields

$\vec{E} = \int \frac{dq}{r^2} \hat{r}$ , so by superposition need only consider point charge

e.g. point charge at the origin of a sphere, radius R



$$\oint \vec{E} \cdot d\vec{a} = \oint \frac{q}{R^2} da = \left(\frac{q}{R^2}\right) 4\pi R^2 = 4\pi q$$



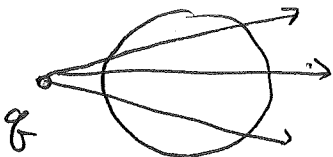
Deformed sphere

$$d\vec{a} = \hat{e}_r da \quad d\vec{a}' = \hat{n}' da' = \hat{n}' \frac{da}{\cos\theta}$$

$$\Rightarrow \vec{E} \cdot d\vec{a}' = \underbrace{(\vec{E} \cdot \hat{n})}_{E \cos\theta} \frac{da}{\cos\theta} = E \cdot da$$

$$\Rightarrow \boxed{\oint \vec{E} \cdot d\vec{a} = 4\pi Q_{enc}}$$

Gauss' Law (integral form)



$$\oint \vec{E} \cdot da = 0$$

$$Q_{enc} = \int_V \rho(\vec{x}) d^3x$$

$$\oint \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) d^3x$$

$$\Rightarrow \boxed{\nabla \cdot \vec{E} = 4\pi \rho(\vec{x})}$$

Gauss' Law (differential form)



• Using Gauss' Law to determine  $\vec{E}$

• Requires very symmetric situation (1) Know form of  $\vec{E}$   
 (2)  $|\vec{E}|$  constant on  $S$

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = E(\xi) \oint da = E(\xi) A(\xi) = 4\pi Q_{enc}$$

Examples: Spheres  $\downarrow$  points, infinite cylinders  $\downarrow$  lines, infinite slabs  $\downarrow$  surfaces

• Sphere  $\vec{E} = E(r)\vec{e}_r$



$$\oint \vec{E} \cdot d\vec{a} = E(r) 4\pi r^2 = 4\pi Q_{enc}$$

$$E(r) = \frac{Q_{enc}}{r^2} \quad (\text{like point charge at origin})$$

• Cylinder ( $\lambda = \text{charge/length}$ )  $\vec{E} = E(\rho)\vec{e}_\rho$

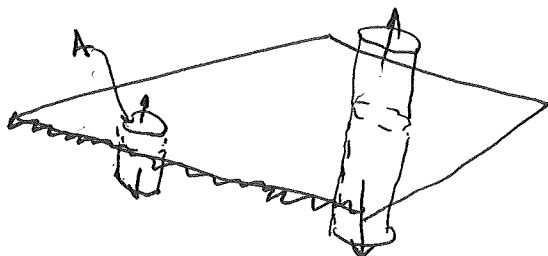


$$\oint \vec{E} \cdot d\vec{a} = E(\rho) (2\pi \rho L) = 4\pi Q_{enc}$$

$$\Rightarrow E(\rho) = \frac{2\lambda}{\rho}$$

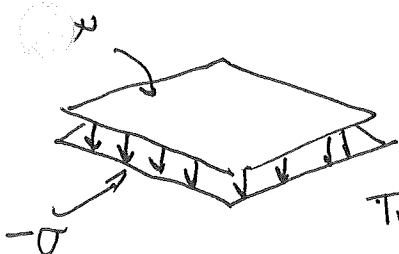
• Slab/surface ( $\sigma = \text{charge/area}$ )  $E = E(z)\vec{e}_z$

$$E(z) = -E(-z)$$



$$\oint \vec{E} \cdot d\vec{a} = 2EA = 4\pi Q_{enc}$$

$$\Rightarrow \boxed{E = 2\pi\sigma} \quad \text{Uniform}$$



Two opposite infinite plane  $E = \begin{cases} 4\pi\sigma & \text{inside} \\ 0 & \text{outside} \end{cases}$

