Physics 511 - Lecture #4

Review Electrostatics II

Paradox: Consider electric field of a point charge

$$\vec{E} = \frac{q}{r^2}\vec{e}_r = \frac{q}{|\vec{x}|^3}\vec{x}$$

$$\vec{\nabla}\cdot\vec{E} = \partial_i E_i = q\,\partial_i\left(\frac{x_i}{r^3}\right) = q\left(\frac{1}{r^3}\underbrace{\partial_i x_i}_{3} + x_i\,\partial_i\left(\frac{1}{r^3}\right)\right)$$

$$= q\left(\frac{3}{r^3} + x_i\left(-\frac{3}{r^4}\right)\partial_i r\right)$$

[Aside $\partial_i r = \frac{x_i}{r^2}$] $\Rightarrow \vec{\nabla}\cdot\vec{E} = q\left(\frac{3}{r^3} - 3\frac{x_i x_i}{r^5}\right) = 0$

But $\int(\vec{\nabla}\cdot\vec{E})\,dV = \oint_S \vec{E}\cdot d\vec{a} = 4\pi q$?

Resolution: $\vec{E}$ discontinuous at the origin! (more later)

Other field equation $\vec{\nabla}\times\vec{E}$

Consider field of point charge: $(\vec{\nabla}\times\vec{E})_i = \epsilon_{ijk}\,\partial_j\left(\frac{x_k}{r^3}\right)$

$$\Rightarrow (\vec{\nabla}\times\vec{E})_i = \epsilon_{ijk}\left(\frac{1}{r^3}\partial_j(x_k) + x_k\,\partial_j(r^{-3})\right)$$

$$= \epsilon_{ijk}\left(\frac{1}{r^3}\delta_{jk} - 3x_k x_j/r^5\right)$$

$$= \underbrace{\epsilon_{ijj}}_{=0}\left(\frac{1}{r^3}\delta_{jk}\right) - 3\underbrace{\frac{(\vec{x}\times\vec{x})_i}{r^5}}_{0} = 0$$

$$\therefore \boxed{\vec{\nabla}\times\vec{E} = 0} \Rightarrow \boxed{\oint_C \vec{E}\cdot d\vec{l} = 0}$$

Electrostatic forces are "conservative"

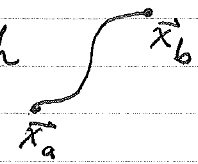
Electrostatic potential $\phi(\vec{x})$

$$\vec{\nabla}\times\vec{E}=0 \quad\Rightarrow\quad \boxed{\vec{E}=-\vec{\nabla}\phi}$$ convention

Equipotential: Surface of constant ϕ

$\vec{E}\perp$ to equipotential

Given $\vec{E}$: Find $\phi(\vec{x})$ by integrating over path



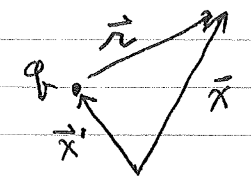
$$\int_{\vec{x}_a}^{\vec{x}_b}\vec{E}\cdot d\vec{\ell} = -\int_{\vec{x}_a}^{\vec{x}_b}\vec{\nabla}\phi\cdot d\vec{\ell} = \phi(\vec{x}_a)-\phi(\vec{x}_b)$$ (independent of path)

Only potential differences are physical.

Define "ground" $\phi(\vec{x}_g)=0$. Typically $\vec{x}_g\Rightarrow\infty$ if no charges there

$$\Rightarrow\quad \boxed{\phi(\vec{x}) = -\int_{\vec{x}_g}^{\vec{x}}\vec{E}\cdot d\vec{\ell}}$$

Example: Point charge at $\vec{x}'$



Path along $\vec{r}$ $\quad d\vec{\ell}=\hat{r}\,dr$

take ground at ∞

$$\phi(\vec{x}) = -\int_{\infty}^{r}\frac{q}{r^2}\,dr = \frac{q}{r} = \frac{q}{|\vec{x}-\vec{x}'|}$$

For a continuous charge distribution (in confined region of space)

$$\boxed{\phi(\vec{x}) = \int\frac{dq(\vec{x}')}{|\vec{x}-\vec{x}'|} = \int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}}$$

Much simpler than integral expression for $\vec{E}(\vec{x})$

Formal solution to field equations

Electrostatics: $\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$ $\boxed{\vec{\nabla}\times\vec{E} = 0}$

Solving, introduce potential

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi$$

$$\therefore \quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \boxed{\nabla^2\phi = -4\pi\rho}$$ Poisson's equation

Solution:

$$\phi(\vec{x}) = \phi_{\text{Homogenous}}(\vec{x}) + \phi_{\text{Particular}}(\vec{x})$$

Homogeneous: $\boxed{\nabla^2\phi_{\text{Hom}} = 0}$: Laplace's equation

ϕ_{Hom} determined by boundary conditions (Jackson 2 and 3)

To find $\phi_{\text{Particular}}$ use superposition principle

Define Green's function: Solution for a unit point charge at $\vec{x}'$

$$\nabla^2 G(\vec{x}-\vec{x}') = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}') \qquad G(\vec{y}) \to 0 \text{ as } |\vec{y}| \to \infty$$

Arbitrary charge distribution is a superposition of point charges

$$\rho(\vec{x}) = \int d^3x'\,\rho(\vec{x}')\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$\Downarrow$$

$$\boxed{\phi(\vec{x}) = \int d^3x'\,\rho(\vec{x}')\,G(\vec{x}-\vec{x}')}$$

Convolution of the source with the Green's function ("impulse response")

Solution $\boxed{G(\vec{x}-\vec{x}') = \dfrac{1}{|\vec{x}-\vec{x}'|}}$ Green's function for a free point charge

Solve for Green's function using Fourier transform

$$G(\vec{y}) = \int \frac{d^3k}{(2\pi)^3} \, \tilde{G}(\vec{k}) \, e^{i\vec{k}\cdot\vec{y}}$$

$$\tilde{G}(\vec{k}) = \int d^3y \; G(\vec{y}) \, e^{-i\vec{k}\cdot\vec{y}}$$

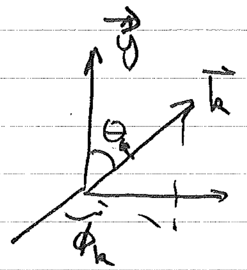
$$\nabla^2 G(\vec{y}) \iff -k^2 \tilde{G}(\vec{k})$$

$$\delta^{(3)}(\vec{y}) \iff 1$$

$$\therefore \quad \tilde{G}(\vec{k}) = \frac{4\pi}{k^2}$$

$$\Rightarrow G(\vec{y}) = \frac{1}{2\pi^2} \int \frac{d^3k}{k^2} \, e^{+i\vec{k}\cdot\vec{y}}$$

Use spherical coordinates. Take polar axis along $\vec{y}$



$$\vec{k}\cdot\vec{y} = k|\vec{y}|\cos\theta_k$$

$$d^3k = k^2 \sin\theta_k \, dk \, d\theta_k \, d\phi_k$$

$$= k^2 \, dk \, d(\cos\theta_k) \, d\phi_k$$

$$\mu \equiv \cos\theta_k \qquad -1 \le \mu \le 1$$

$$\Rightarrow G(\vec{y}) = \frac{2\pi}{2\pi^2} \int_0^\infty dk \, \underbrace{\int_{-1}^{1} d\mu \, e^{i\mu k|\vec{y}|}}_{2\frac{\sin k|\vec{y}|}{k|\vec{y}|}}$$

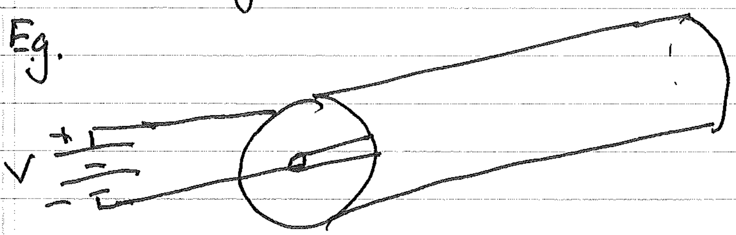
$$\Rightarrow G(\vec{y}) = \frac{2}{\pi} \frac{1}{|\vec{y}|} \underbrace{\int_0^\infty d\xi \, \frac{\sin\xi}{\xi}}_{\pi/2}$$

$$\Rightarrow \boxed{G(\vec{y}) = \frac{1}{|\vec{y}|}} \qquad \boxed{G(\vec{x}-\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}}$$

In principle, if we know $\rho(\vec{x})$ <u>everywhere</u> we can ~~aov~~ find $\phi(\vec{x})$ and hence $\vec{E}(\vec{x})$

In many situations this is not the case

Eg.

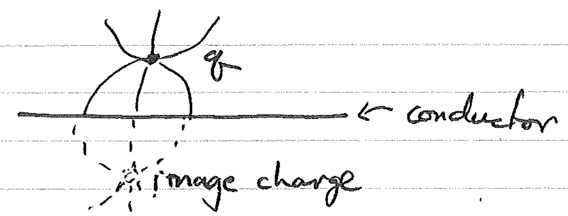


We know $\phi(\vec{x})$ on <u>boundary</u>. Charges adjust to satisfy b.c.

<u>Uniqueness theorem:</u> Given $\rho(\vec{x})$ and $\phi(\vec{x})$ or $\hat{n}\cdot\vec{\nabla}\phi$ everywhere on a surface ~~sour~~ surrounding a region $\Longrightarrow$ $\phi(\vec{x})$ is uniquely determined inside the region

<u>Solution methods</u>

(1) Method of images



(2) Orthogonal function expansion (Most of Jackson chap 2 + 3)

(3) Conformal mapping (2D problems)

(4) Numerical methods

General solution: Include boundary effect in Green's function

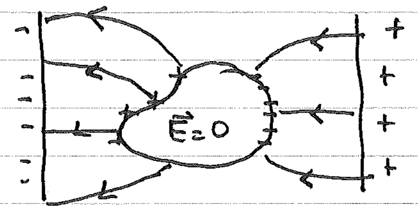
$$G(\vec{x}-\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \qquad \nabla^2 F(\vec{x},\vec{x}') = 0$$

$$\boxed{\phi(\vec{x}) = \int_V \rho(\vec{x}')\, G(\vec{x}-\vec{x}')\, d^3x' + \oint_S \left(G(\vec{x}-\vec{x}') \frac{\partial \Phi}{\partial n} - \phi(\vec{x}') \frac{\partial G}{\partial n'} \right)}$$

L4.5

Conductors: Ideal case – infinitely mobile reservoir of free, non-interacting charges (infinite conductivity σ_c)

- Conductor in an external field



(i) $\vec{E} = 0$ inside conductor

(ii) Net free charge density resides on surface $\sigma_s(\vec{x})$

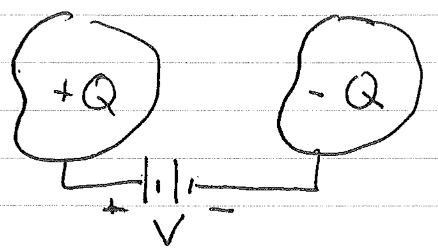
(iii) On surface $\vec{E}_{\parallel} = 0$

$\Rightarrow$ Surface is equipotential

(iv) $\Delta \vec{E}_{\perp} = 4\pi\sigma_s(\vec{x}) \Rightarrow \vec{E}_{\perp}(\vec{x}) = 4\pi\sigma_s(\vec{x})$

Boundary condition

- Capacitance: Given conductor fixed at potential difference V



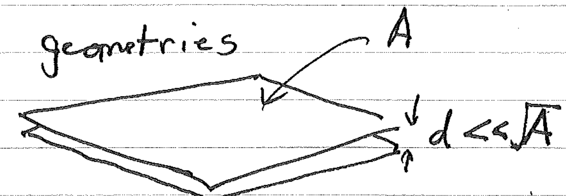
Induced charge purely a function of geometry

$$Q = CV$$

$\nwarrow$ Capacitance

Finding C for simple geometries

e.g. Parallel plates



Assume charge $\pm Q$. Simple case σ_s uniform

$\Rightarrow$ Find $\vec{E}$ $\qquad E = 4\pi\sigma_s = 4\pi\frac{Q}{A}$

Find V: Here $V = \int_-^+ E\,dl = Ed \Rightarrow E = \frac{V}{d}$

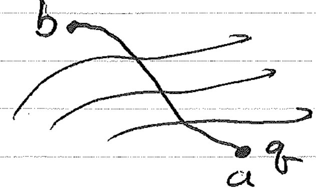
$$\Rightarrow \boxed{C = \frac{Q}{V} = \frac{A}{4\pi d}}$$

In c.g.s $[C] = \text{cm}$

S.I $[C] = \text{farad}$

Energy and work in electrostatics

- Charges in an external field



$\vec{E}$ (due to other charges)

Work done moving charge from a to b against $\vec{E}$

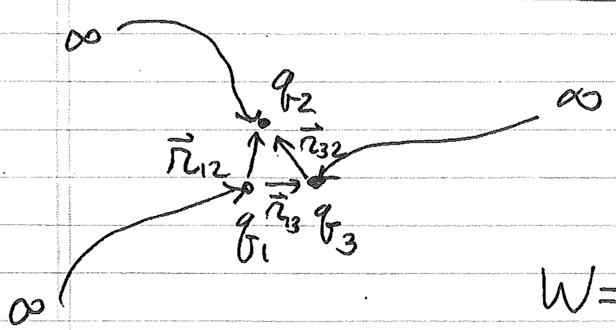
$$W = \int_a^b \vec{F}_{\text{I exert}} \cdot d\vec{l} = -\int_a^b \vec{F}_{\text{Field}} \cdot d\vec{l} = -q \int_a^b \vec{E} \cdot d\vec{l}$$

$$= q\left(\phi(b) - \phi(a)\right) = \Delta U_{ba}$$ change in potential energy of q

$\Rightarrow \quad q\,\phi(\vec{x})$ = Potential energy of q in the external field (relative to ground)

$$U = \int d^3x \; \rho(\vec{x})\, \phi_{\text{external}}(\vec{x})$$

- Work done to assemble a charge distribution



$$W = \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_2}{r_{23}}$$

$$W = \frac{1}{2} \sum_{i \neq j}' \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i' q_i \underbrace{\sum_{j \neq i} \frac{q_j}{r_{ij}}}_{\phi(\vec{x}_i)}$$

no double counting

self potential $\rightarrow \phi(\vec{x}_i)$

$$\Rightarrow \boxed{W = \frac{1}{2} \sum_i q_i\, \phi(\vec{x}_i) = \left(\frac{1}{2}\right) \int d^3x \; \rho(\vec{x})\, \phi_{\text{self}}(\vec{x}) = U}$$

self energy

U : potential energy of charge distribution

Potential energy stored in the electric field

$$\rho(\vec{x}) = \frac{\vec{\nabla}\cdot\vec{E}}{4\pi} \Rightarrow U = \frac{1}{8\pi}\int_{\text{all space}} d^3x \,(\vec{\nabla}\cdot\vec{E})\,\phi(\vec{x})$$

Integration by parts: $\vec{\nabla}\cdot(\phi\vec{E}) = \phi(\vec{\nabla}\cdot\vec{E}) + \vec{E}\cdot\vec{\nabla}\phi$

$$\Rightarrow (\vec{\nabla}\cdot\vec{E})\,\phi(\vec{x}) = -\vec{\nabla}\phi\cdot\vec{E} - \phi\,\vec{\nabla}\cdot\vec{E} = \vec{E}\cdot\vec{E} - \vec{\nabla}\cdot(\phi\vec{E})$$

$$\Rightarrow U = \frac{1}{8\pi}\int_{\text{all space}} d^3x\,\vec{E}\cdot\vec{E} + \frac{1}{8\pi}\int_{\text{all space}} d^3x\,\vec{\nabla}\cdot(\phi\vec{E})$$

Aside: $\int_{\text{all space}} d^3x\,\vec{\nabla}\cdot(\phi\vec{E}) = \lim_{S\to\infty}\oint(\phi\vec{E})\cdot d\vec{a} = 0$

(assuming $(\phi\vec{E}) \to 0$ at ∞ faster than $\frac{1}{r^2}$)

$$\therefore \boxed{U = \frac{1}{8\pi}\int_{\text{all space}} (\vec{E}\cdot\vec{E})\,d^3x}$$

Work I do in assembling charge distribution converted to energy of the field

Note: Point charge $U = \frac{1}{8\pi}\int_0^\infty \frac{q}{r^2}(4\pi r^2)\,dr = \infty$?

When expressed in terms of continuous distribution U includes energy to build up a point charge $\Rightarrow \infty$

- Energy in a capacitor: (charge build up)

Given charge q already on plates, potential energy to add dq

$$dU = dq\,V = dq\,\frac{q}{C} \Rightarrow U = \int_0^Q dq\,\frac{q}{C}$$

$$\Rightarrow \boxed{U = \frac{Q^2}{2C} = \frac{1}{2}CV^2} = \frac{1}{2}\left(\frac{A}{4\pi d}\right)V^2 = \frac{1}{8\pi}E^2(Ad)$$