Physics 511 Lecture #5

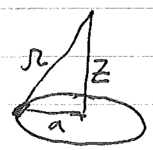
Multipole Expansions I

- Generally, given $\rho(\vec{x})$ confined to a compact region R

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}$$

Cannot solve in general. Seek approximation scheme for $|\vec{x}| >> R$

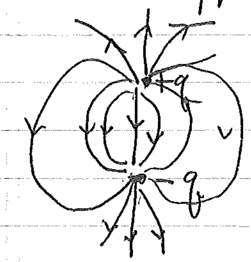
Examples: • Ring of charge, uniform

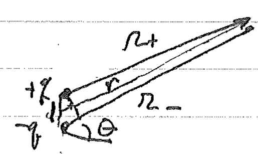


$$\phi(z) = \int \frac{dq}{r} = \frac{Q}{\sqrt{z^2+a^2}} = \frac{Q}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$

$$z >> a \qquad \phi(z) \cong \underbrace{\frac{Q}{z}}_{\text{point charge}} - \underbrace{\frac{Qa^2}{2z^3}}_{\text{correction term}} + + \cdots$$

• Two oppositely charged points



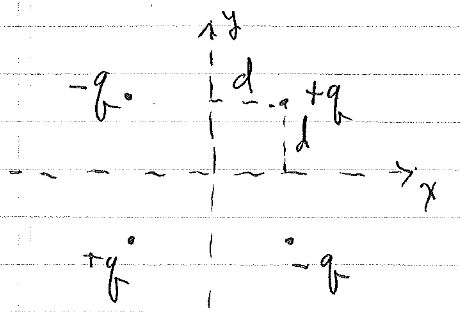


$$\phi(\vec{x}) = \frac{q}{r_+} - \frac{q}{r_-} \approx \frac{q}{r_+ r_-}(r_+ - r_-)$$

$$\Rightarrow \phi(\vec{x}) \cong q\frac{d\cos\theta}{r^2}$$

Dipole

• Two opposite dipoles

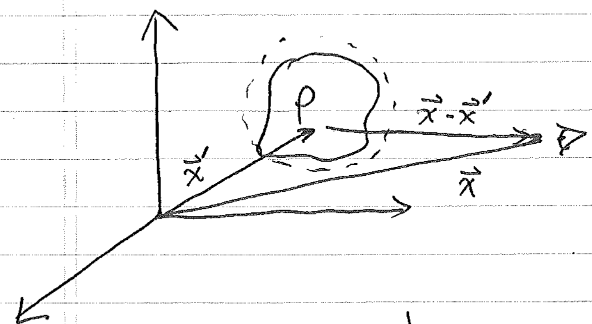


$$\phi(\vec{x}) = \frac{q}{\sqrt{(x-d)^2+(y-d)^2+z^2}} - \frac{q}{\sqrt{(x-d)^2+(y+d)^2+z^2}} + \cdots$$

$$\approx 12qd^2\frac{xy}{r^5} \sim \frac{1}{r^3}$$

Quadrapole

General multipole expansion



Localized charge distribution

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}$$

Far from distribution $\frac{|\vec{x}'|}{|\vec{x}|} = \frac{r'}{r} \ll 1$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{\left(r^2 - 2\vec{x}\cdot\vec{x}' + r'^2\right)^{1/2}} = \frac{1}{r}\left(1 - 2\hat{r}\cdot\hat{r}'\left(\frac{r'}{r}\right) + \left(\frac{r'}{r}\right)^2\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 - 2\hat{r}\cdot\hat{r}'\,\delta + \delta^2\right)^{-1/2} \qquad \delta \equiv \frac{r'}{r} \ll 1$$

Recall binomial expansion: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \mathcal{O}(x^3)$

$$\Rightarrow (1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \mathcal{O}(x^3)$$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r}\left(1 - \tfrac{1}{2}\left(-2\hat{r}\cdot\hat{r}'\delta + \delta^2\right) + \tfrac{3}{8}\left(-2\hat{r}\cdot\hat{r}'\delta + \delta^2\right)^2 + \cdots\right)$$

Keep terms only to $\mathcal{O}(\delta^2)$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r}\left(1 + \hat{r}\cdot\hat{r}'\delta - \tfrac{1}{2}\delta^2 + \tfrac{3}{2}(\hat{r}\cdot\hat{r}')^2\delta^2 + \mathcal{O}(\delta^3)\right)$$

$$= \frac{1}{r}\left(1 + \hat{r}\cdot\hat{r}'\left(\frac{r'}{r}\right) + \tfrac{1}{2}\left(3(\hat{r}\cdot\hat{r}')^2 - 1\right)\left(\frac{r'}{r}\right)^2 + \cdots\right)$$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2} + \frac{1}{2}\,\frac{1}{r^3}\,\hat{r}\cdot\big(\underbrace{3\vec{x}'\vec{x}' - \overleftrightarrow{\mathbb{1}}r'^2}_{\text{Tensor}}\big)\,\hat{r} + \cdots$$

Plugging back into expression for $\phi(\vec{x})$

$$\phi(\vec{x}) = \frac{1}{r}\int d^3x'\,\rho(\vec{x}') + \frac{\hat{r}}{r^2}\cdot\int d^3x'\,\vec{x}'\,\rho(\vec{x}')$$

$$+\frac{1}{2}\,\hat{r}\cdot\frac{\int d^3x'\,(3\vec{x}'\vec{x}' - r'^2\,\overleftrightarrow{\mathbb{1}})\,\rho(\vec{x}')}{r^3}\cdot\hat{r} + \cdots$$

$$\phi(\vec{x}) = \frac{q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2} + \frac{1}{2}\,\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3} + \cdots + \frac{1}{n!}\,\frac{\hat{r}_{i_1}\hat{r}_{i_2}\cdots\hat{r}_{i_n}Q_{i_1\cdots i_n}}{r^n}$$

$q = \int d^3x\,\rho(\vec{x})$ (monopole moment) <u>Scalar</u>

$\vec{p} = \int d^3x\,\vec{x}\,\rho(\vec{x})$ (dipole moment) <u>vector</u>

$\overleftrightarrow{Q} = \int d^3x\,(3\vec{x}\,\vec{x} - \overleftrightarrow{\mathbb{1}}\,r^2)\,\rho(\vec{x})$ (quadrapole moment) <u>2-rank tensor</u>

Component form $Q_{ij} = \int d^3x\,(3x_ix_j - \delta_{ij}r^2)\,\rho(\vec{x})$

Matrix
$$\underset{\approx}{Q} = \int d^3x \begin{bmatrix} 2x^2-y^2-z^2 & 3xy & 3xz \\ & 2y^2-x^2-z^2 & 3yz \\ & & 2z^2-x^2-y^2 \end{bmatrix} \rho(x,y,z)$$

Symmetric $Q_{ij} = Q_{ji}$

Traceless $\mathrm{Tr}(Q_{ij}) = 0$ "irreducible"

Physical meaning

Let $\rho(\vec{x}) = q_+ \underbrace{N_+(\vec{x})} - q_- \underbrace{N_-(\vec{x})}$

Distribution of positive/negative charge

$$\int N_\pm(\vec{x})\, d^3x = 1 \quad \text{normalized}$$

Monopole $\Rightarrow$ $q = q_+ - q_-$: Net charge

Dipole $\Rightarrow$ $\vec{p} = q_+ \underbrace{\int \vec{x}\, N_+(\vec{x})\, d^3x}_{\langle \vec{x} \rangle_+} - q_- \underbrace{\int \vec{x}\, N_-(\vec{x})\, d^3x}_{\langle \vec{x} \rangle_-}$

$\vec{p} = \left(\text{Ave position of positive charge}\right) - \left(\text{Average position of negative charge}\right)$

Quadrapole $\overleftrightarrow{Q} = \overleftrightarrow{Q}_+ - \overleftrightarrow{Q}_-$

$$\overleftrightarrow{Q}_+ = q_+ \int \left(3\vec{x}\,\vec{x} - r^2 \overleftrightarrow{\mathbb{1}}\right) N_+(\vec{x})\, d^3x \quad \left(\text{Something about "spread" of charge about origin}\right)$$

Suppose $N_+(\vec{x})$ is spherically symmetric about origin $= N_+(r)$

Off diagonal components: $Q_{ij} = 3q_+ \int x_i x_j\, N_+(\vec{x})\, d^3x = 0$ (after integration over

Diagonal components: say x

$$Q_{xx} = q_+ \Big[3 \underbrace{\int x^2 N_+(r)\, d^3x}_{\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle} - \underbrace{\int r^2 N_+(r)\, d^3x}_{\langle r^2 \rangle} \Big] = 0$$

$\Rightarrow$ For a single species of charge $\overleftrightarrow{Q}$ measures the deviation of the distribution from spherical

(Next Page)

L # 5.5

<u>Quadrapole contenued</u>

Note: $Q_{ij} = Q_{ji}$ and $\mathrm{Tr}(Q_{ij}) = 0$.

$\Rightarrow$ Generally need 5 numbers to specify $\overleftrightarrow{Q}$

Case of ~~[illegible]~~ cylindrical symmetry about z

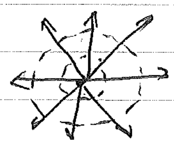
$$Q_{xy} = Q_{xz} = Q_{yz} = 0 \quad \text{(no off diagonal elements)}$$

$$Q_{xx} = Q_{yy} \qquad \mathrm{Tr}(\underset{\approx}{Q}) = 0 \quad \Rightarrow \quad Q_{xx} = Q_{yy} = -\tfrac{1}{2} Q_{zz}$$

Need specify only <u>one number</u>: "The quadropole moment" Q

<u>Multipole potentials (contour plots)</u>

- <u>Monopole</u>: $\Phi^{(0)}(\vec{x}) = \dfrac{q}{r}$ E-field $-\vec{\nabla}\Phi^{(0)} \sim \dfrac{1}{r^2}$

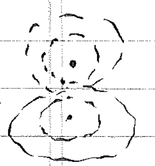


Spherically symmetric contours

- <u>Dipole</u>: $\Phi^{(1)}(\vec{x}) = \vec{p}\cdot\dfrac{\hat{r}}{r^2} = \dfrac{\vec{p}\cdot\vec{x}}{r^3} = \dfrac{p_i x_i}{r^3}$

$$= p_x\left(\frac{x}{r^3}\right) + p_y\left(\frac{y}{r^3}\right) + p_z\left(\frac{z}{r^3}\right)$$

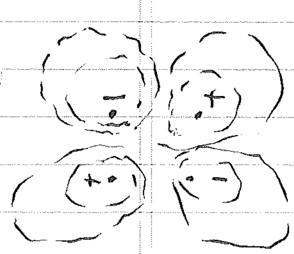
Two poled equipotential



$$\vec{E}^{(1)} \sim \frac{1}{r^3}$$

- <u>Quadrapole</u>: $\Phi^{(2)}(\vec{x}) = \dfrac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} = \dfrac{1}{2} Q_{ij}\dfrac{x_i x_j}{r^5}$

$$= \frac{1}{2r^5}\left(x^2 Q_{xx} + y^2 Q_{yy} + z^2 Q_{zz} + 2xy\,Q_{xy} + 2xz\,Q_{xz} + 2yz\,Q_{yz}\right)$$



$$\vec{E}^{(2)} \sim \frac{1}{r^4}$$

Examples: Calculating the multipole moments

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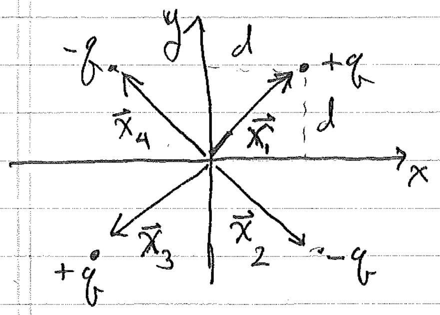

$$q_{Net} = 0$$

$$\vec{p} = \sum_\alpha \vec{x}_\alpha q_\alpha = 0$$

$$q_+ \langle \vec{x}_+ \rangle = q_- \langle \vec{x}_- \rangle = 0$$

$$Q_{ij} = \sum_\alpha q^{(\alpha)} \left(3 x_i^{(\alpha)} x_j^{(\alpha)} - r^{2\,(\alpha)} \delta_{ij}\right) \qquad r^{2\,(\alpha)} = 2d^2$$

$Q_{xz} = Q_{yz} = Q_{zz} = 0$ (everything in x-y plane)

$$\mathrm{Tr}\,(Q_{ij}) = 0 \quad \Rightarrow \quad Q_{xx} = -Q_{yy}$$

$$Q_{xx} = \underbrace{q(3d^2 - 2d^2)}_{\text{particle } 1} - \underbrace{q(3d^2 - 2d^2)}_{2} + \underbrace{q(3d^2 - 2d^2)}_{3} - \underbrace{q(3d^2 - 2d^2)}_{4} = 0$$

(Positive and negative distributions about x-axis are same)

$$Q_{xy} = \sum_\alpha 3 q^{(\alpha)} x^{(\alpha)} y^{(\alpha)} = 3q\left\{d^2 - (d)(-d) + (-d)(-d) - (-d)(d)\right\}$$

$$= 12 q d^2 = Q_{yx} \qquad \Rightarrow \underset{\approx}{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 12 q d^2$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{2r^5}\left(xy\, Q_{xy} + yx\, Q_{yx}\right) = \frac{xy}{r^5} Q_{xy} = 12 q d^2 \frac{xy}{r^5}$$

(Same as from the Taylor series expansion of the exact potential)

L#5.7

• Ring of charge $\rho(\vec{x}) = \frac{q}{2\pi a}\,\underline{\delta(\rho - a)\,\delta(z)}$

$$\Rightarrow q = \int \rho(\vec{x})\, d^3x = \frac{q}{2\pi}\int_0^{2\pi} d\varphi = q \quad \checkmark$$

$$\Rightarrow \vec{P} = \int \vec{x}\,\rho(\vec{x})\, d^3x = \int a\,\vec{e}_\rho \left(\frac{q}{2\pi a}\right) a\,d\varphi = \frac{qa}{2\pi}\int \hat{e}_\rho\, d\varphi = 0$$

$$q_+ \langle \vec{x} \rangle_+ = 0$$

$\Rightarrow \overleftrightarrow{Q} = ?$ Symmetry $Q_{xy} = Q_{yz} = Q_{xz} = 0$

$$Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$$

$$Q_{zz} = \int \underbrace{(3z^2 - r^2)}_{(2z^2 - \rho^2)}\, \rho(r,z) \; 2\pi\rho\, d\rho\, dz$$

$$\rho(r,z) \rightarrow \left(\frac{q}{2\pi a}\,\delta(\rho - a)\,\delta(z)\right)$$

$= -qa^2$ (all charge distributed at radius a)

$$\Rightarrow \phi(\vec{x}) = \frac{q}{r} + \frac{Q_{zz}}{2r^5}\left(-\frac{(x^2+y^2)}{2} + z^2\right) + \cdots$$

On z-axis $x = y = 0$ $\quad r = z$

$$\phi(z) = \frac{q}{z} - \frac{qa^2}{2z^3} + \cdots$$

Same as we obtained from the Taylor series expansion of the exact potential on axis