

Physics 511 - Lecture #7

Magnetostatics I

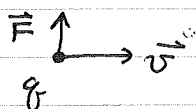
Lorentz Force:

$$\vec{F} = q \frac{\vec{v}}{c} \times \vec{B}$$

c speed of light



Neutral current carrying wire



$$c = 3 \times 10^{10} \text{ cm/s}$$

Magnetic field exerts force on moving charges: currents
(Reference frame?)

Currents are the source of \vec{B} -field

Generally

Single point charge $I = \frac{dq}{dt}$

$$dq = \rho da \vec{v} \cdot \hat{n} dt$$

$$dI = \frac{dq}{dt} = \rho \vec{v} \cdot \hat{n} da$$

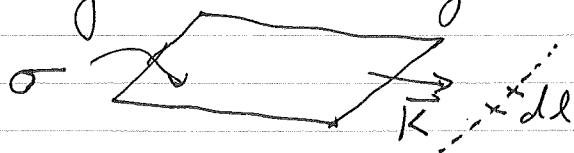
Current density $\vec{J} = \sum_a q_a \delta(\vec{x} - \vec{x}_a) \vec{v}_a$
 $= \rho \vec{v}$ (for uniform flow)

$$I = \int_S \vec{J} \cdot d\vec{a}$$

flux of current through surface

$$[\vec{J}] = \frac{\text{current}}{\text{Area}}$$

Flowing surface charge: $\vec{K} = \sigma \vec{v}$



Flux through line

$$I = \int_C \vec{K} \cdot d\vec{l}$$

Flowing line charge: $I = \lambda \vec{v}$



Flux through a point

General Lorentz force Law:

Collection of point charges $\vec{F} = \sum_{\alpha} q_{\alpha} (\vec{E}(\vec{x}_{\alpha}) + \frac{\vec{v}_{\alpha}}{c} \times \vec{B}(\vec{x}_{\alpha}))$

\Rightarrow Continuous distribution, introduce delta functions

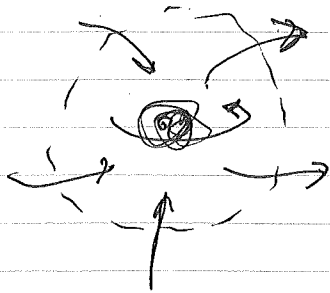
$$\vec{F} = \int d^3x \left[\left(\sum_{\alpha} q_{\alpha} \delta^{(3)}(\vec{x} - \vec{x}_{\alpha}) \right) \vec{E}(\vec{x}) + \frac{1}{c} \left(\sum_{\alpha} q_{\alpha} \vec{v}_{\alpha} \delta^{(3)}(\vec{x} - \vec{x}_{\alpha}) \right) \times \vec{B}(\vec{x}) \right]$$

$$\Rightarrow \boxed{\vec{F} = \int d^3x \left(\rho(\vec{x}) \vec{E}(\vec{x}) + \frac{1}{c} \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) \right)}$$

Lexicon: $\int d^3x$ \rightarrow $\vec{K} da$ \rightarrow $I d\vec{l}$ \rightarrow $q \delta^{(3)}$ (Use delta function)
 volume current surface current line current \nearrow point charge

Conservation of charge and steady currents

Change inside change because of flow



$$\frac{dQ_{\text{inside}}}{dt} = - (\text{Flux out of surface})$$

$$\frac{d}{dt} \int_V \rho d^3x = - \oint_S \vec{J} \cdot d\vec{a}$$

Divergence theorem $\Rightarrow \int_V \frac{\partial \rho}{\partial t} d^3x = \int_V (-\vec{\nabla} \cdot \vec{J}) d^3x$

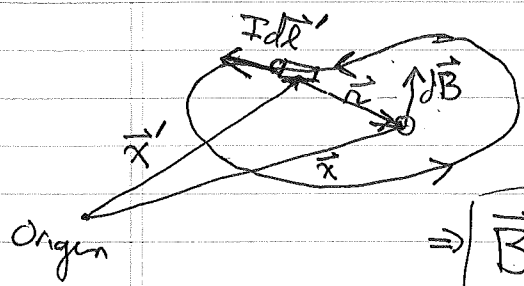
$$\therefore \boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}}$$

Charge "continuity equation"
 local density charge density changes if current diverges

Magnetostatics $\vec{\nabla} \cdot \vec{J} = 0 \equiv \frac{\partial \rho}{\partial t} = 0$: Electrostatics
 Steady current

Currents are the source of \vec{B}

Steady current \Rightarrow Biot-Savart (empirical)

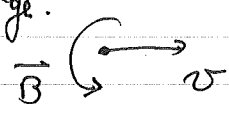


$$d\vec{B} = \frac{1}{c} \frac{I d\vec{l}' \times \hat{r}}{r^2} = \frac{I d\vec{l}' \times (\vec{x} - \vec{x}')}{c |\vec{x} - \vec{x}'|^3}$$

$$\Rightarrow \vec{B} = \frac{1}{c} \int_{\text{line}} I d\vec{l}' \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \frac{1}{c} \int_{\text{surface}} \vec{K} da' \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

$$\vec{B} = \frac{1}{c} \int_{\text{volume}} d^3x' \vec{J}(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$

Point charge:

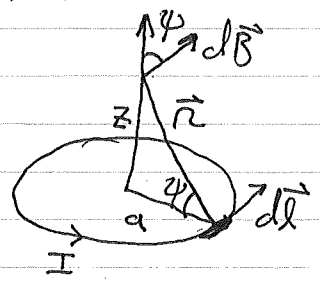


$$\vec{B} \stackrel{?}{=} \frac{1}{c} q \vec{v} \times \hat{r} / r^2$$

No: Not steady current

Only approximate for $v \ll c$

Example: Current loop, steady current I



Only z-component survives on axis

$$dB_z = |d\vec{B}| \cos \psi = \frac{I dl}{c r^2} \cos \psi$$

$$dl = a d\phi, \cos \psi = \frac{a}{r}, r = \sqrt{z^2 + a^2}$$

$$\Rightarrow \vec{B}(z) = \left(\oint dB_z \right) \vec{e}_z = \oint \frac{I a d\phi}{c (z^2 + a^2)^{3/2}} \vec{e}_z$$

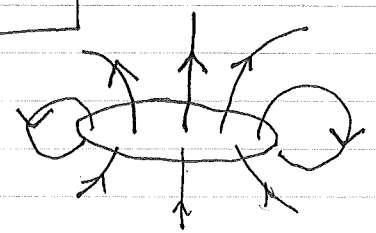
on axis \Rightarrow

$$\vec{B}(z) = \frac{2\pi a^2 I}{c (z^2 + a^2)^{3/2}} \vec{e}_z$$

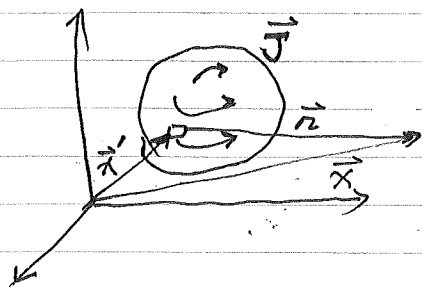
lim $z \gg a$

$$\vec{B} \rightarrow \frac{2\pi a^2 I}{c z^3} \vec{e}_z$$

Dipole field



Field equations:



$$\vec{B} = \frac{1}{c} \int d^3x' \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

$$= \frac{1}{c} \int d^3x' \vec{J}(\vec{x}') \times \left(-\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} \right)$$

Aside: $\vec{\nabla}_x \left(\vec{\nabla} f(\vec{x}) \right) = -\vec{\nabla}_x \vec{\nabla} f(\vec{x})$
constant

$$\therefore \vec{B}(\vec{x}) = \vec{\nabla}_x \left(\frac{1}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right) = \vec{\nabla}_x \vec{A}$$

$$\vec{A} = \frac{1}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \text{Vector potential}$$

Differential form of magnetostatic field eqns.

$$\vec{B} = \vec{\nabla}_x \vec{A} \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0} \quad \text{No magnetic monopoles}$$

$$\vec{\nabla}_x \vec{B} = \vec{\nabla}_x (\vec{\nabla}_x \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$= \underbrace{\frac{1}{c} \vec{\nabla} \int \vec{J}(\vec{x}') \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|}}_{I_1} - \underbrace{\frac{1}{c} \int \vec{J}(\vec{x}') \nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} d^3x'}_{I_2}$$

$$I_1 = \frac{1}{c} \vec{\nabla} \int \vec{J}(\vec{x}') \cdot \left(-\vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|} \right) \stackrel{\substack{\uparrow \\ \text{Integrate} \\ \text{by parts}}}{=} \frac{1}{c} \vec{\nabla} \left(\int \frac{\vec{J} \cdot \vec{\nabla}'}{|\vec{x} - \vec{x}'|} + \oint_{\text{surface at } \infty} \frac{\vec{J}}{|\vec{x} - \vec{x}'|} \right)$$

$$I_2 = -\frac{1}{c} \int \vec{J}(\vec{x}') (-4\pi \delta^3(\vec{x} - \vec{x}')) = \frac{4\pi}{c} \vec{J}(\vec{x})$$

$$\therefore \boxed{\vec{\nabla}_x \vec{B} = \frac{4\pi}{c} \vec{J}(\vec{x})} \quad \text{Ampère's Law}$$