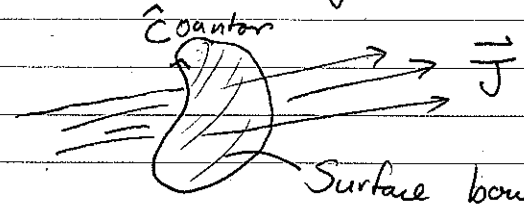
Physics 511 – Lecture #8

Magnetostatics II

Ampère's Law: Integral Form



Flux of $\vec{J}$ through $\mathcal{S}$ $\Rightarrow \int_{\mathcal{S}} \vec{J}\cdot da = I_{enc}$

= Current enclosed by $\mathcal{C}$

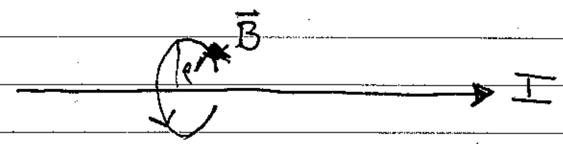
$$\Rightarrow \int \vec{J}\cdot d\vec{a} = \frac{c}{4\pi}\int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} \underset{\substack{\uparrow \\ \text{Stokes theorem}}}{=} \frac{c}{4\pi}\oint_{\mathcal{C}} \vec{B}\cdot d\vec{l}$$

$$\Rightarrow \boxed{\oint_{\mathcal{C}} \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc}}$$

Remember Right Hand Rule

Use Integral form of Ampère's Law like Gauss's Law to find $\vec{B}$ for highly symmetric distributions

- Most basic example: Infinite line current

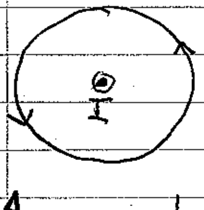


Symmetry

$\vec{B} = B(\rho)\,\vec{e}_\phi$

radial distance from axis

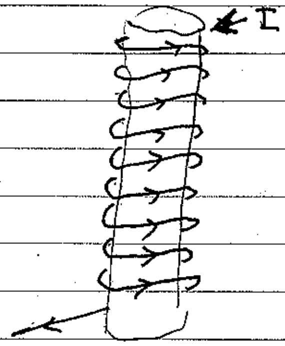
Head on



$$\oint_{\mathcal{C}} \vec{B}\cdot d\vec{l} = B(\rho)\oint_{\mathcal{C}} dl = (2\pi\rho)\,B(\rho) = \frac{4\pi}{c} I$$

"Amperian loop" $\Rightarrow$ $\boxed{\vec{B} = \frac{2I}{c\rho}\,\vec{e}_\phi}$

Eg Infinite Solenoid



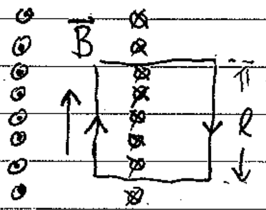
$n = \frac{\#\text{turns}}{\text{length}}$

Using Biot-Savart we can prove that B is uniform inside, zero outside

$\Rightarrow \vec{B} = B_0 \vec{e}_z$ inside

symmetry

Cross section

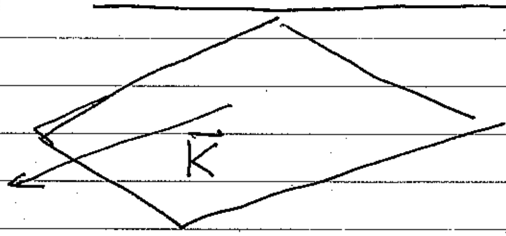


$$\oint \vec{B}\cdot d\vec{l} = Bl = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{a} = \frac{4\pi}{c} N_{\text{turns}} I$$

$$\Rightarrow \boxed{\begin{aligned} \vec{B} &= \frac{4\pi}{c} n I \hat{e}_z \quad \text{inside} \\ &= 0 \quad \text{outside} \end{aligned}}$$

Note $nI = K$ (surface current)

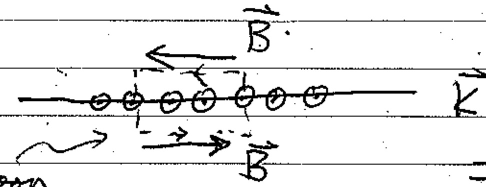
Eg. Planar Sheet of current

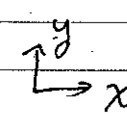


Infinite $\Rightarrow |\vec{B}|$ independent of distance from plane

$\vec{B} \perp$ to $\vec{K}$ and $\perp$ to $\vec{x}$

$\Rightarrow$ Edge on





$\vec{J} = \vec{K}\,\delta(y)$

$$\oint \vec{B}\cdot d\vec{l} = 2lB = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{a} = \frac{4\pi}{c}\int K\,dx = \frac{4\pi}{c}Kl$$

$$\Rightarrow \boxed{B = \frac{2\pi}{c}K}$$

Formal Solution in magnetostatics

Field equations: $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$

$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{J}$

$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}$

Gauge invariance: Can always add an "irrotational" vector field to $\vec{A}$ without changing $\vec{B}$

$\vec{A} \Rightarrow \vec{A}+\vec{\nabla}\Lambda \qquad \vec{B} \Rightarrow \vec{\nabla}\times(\vec{A}+\vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A}$

Gauge transformation $\qquad \vec{B} \Rightarrow \vec{B}$

Fixing the gauge: In magnetostatics choose $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A}=0$ - Always possible

$\therefore$ $\boxed{\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}}$ Poisson's equation for each Cartesian component

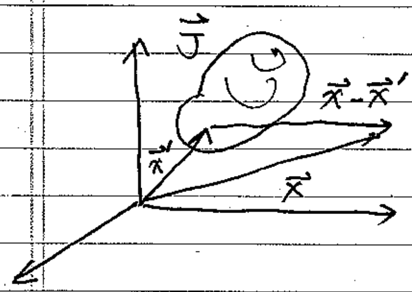
Formal Solution $\boxed{\vec{A} = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$
(assuming $\vec{J}\to 0$ at ∞)

$\boxed{\text{Careful}}$: This is a vector equation so generally, direction of $\vec{J}(\vec{x}')$ depends on $\vec{x}'$

Not do-able for most $\vec{J}$

Need approximations $\Rightarrow$ Multipole expansion

Multipole Expansion in Magnetostatics



In "transverse-gauge"

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

with $\vec{\nabla}\cdot\vec{J}=0$, Localized source

Taylor series: $$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!}\,(\vec{x}'\cdot\vec{\nabla})^{\ell}\,\frac{1}{r}$$

"Monopole Term" $$\vec{A}^{(0)} = \frac{1}{cr}\int d^3x'\,\vec{J}(\vec{x}') = 0$$

Proof: $$\vec{\nabla}'\cdot\left(x_i'\,\vec{J}(\vec{x}')\right) = \partial_k'\left(x_i\,J_k(\vec{x})\right) = \underbrace{\delta_{ik}J_k}_{J_i} + x_i'(\vec{\nabla}'\cdot\vec{J}')$$

$$\Rightarrow \int_V \vec{J}(\vec{x}')\,d^3x' = \int \vec{\nabla}'\cdot(\vec{x}'\,\vec{J}(\vec{x}))\,d^3x = \oint_S (\vec{x}'\,\vec{J}(\vec{x}'))\cdot d\vec{a}$$

as $S\to\infty$ $$\int_V \vec{J}(\vec{x}')\,d^3x' = 0$$

"Dipole term": $$\vec{A}^{(1)}(\vec{x}) = -\frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}')\,(\vec{x}'\cdot\vec{\nabla})\,\frac{1}{r}$$

Tensor notation $$A_i^{(1)}(\vec{x}) = \left(-\frac{1}{c}\int d^3x'\,J_i(\vec{x}')\,x_k'\right)\partial_k\frac{1}{r}$$

Break up into symmetric/antisymmetric pieces

$$J_i x_k' = \frac{1}{2}(J_i x_k' + J_k x_i') + \frac{1}{2}(J_i x_k' - J_k x_i')$$

First term integrates to zero:

$$\vec{\nabla}\cdot(x_i' x_j'\,\vec{J}) = x_j'\,J_i(\vec{x}') + x_i'\,J_j(\vec{x}')$$

$$\therefore\ A_i^{(1)} = -\frac{1}{2c}\int d^3x' \left(x_j' J_i - x_i' J_j\right)\left(\partial_j \frac{1}{r}\right)$$

$$= \frac{1}{2c}\int d^3x'\, \epsilon_{ijk} \left(\vec{x}' \times \vec{J}(\vec{x}')\right)_k \left(\partial_j \frac{1}{r}\right)$$

$$A_i^{(1)} = \epsilon_{ijk}\left[\frac{1}{2c}\int d^3x' \left(\vec{x}' \times \vec{J}(\vec{x}')\right)\right]_k \partial_j \frac{1}{r}$$

$$\Rightarrow\ \boxed{\begin{aligned} &\vec{A}^{(1)} = -\vec{m}\times\vec{\nabla}\frac{1}{r} = \frac{\vec{m}\times\hat{r}}{r^2} \\ &\vec{m} = \frac{1}{2c}\int d^3x'\ \vec{x}'\times\vec{J}(\vec{x}') \quad \text{Magnetic dipole moment} \end{aligned}}$$

For a magnetic dipole:

$$\vec{B}^{(1)} = \vec{\nabla}\times\vec{A} = -\vec{\nabla}\times\left(\vec{m}\times\vec{\nabla}\frac{1}{r}\right) = -\epsilon_{ijk}\epsilon_{klp}\,\partial_j m_l \partial_p \frac{1}{r}$$

$$= -\left(\delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl}\right) m_l \partial_j \partial_p \frac{1}{r}$$

$$= -m_i \underbrace{\left(\partial_j^2 \frac{1}{r}\right)}_{0\ r\neq 0} + m_j \partial_j \partial_i \frac{1}{r} = \partial_i\ \vec{m}\cdot\vec{\nabla}\frac{1}{r}$$

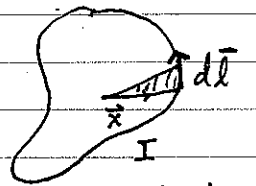
$$= -\partial_i\ \vec{m}\cdot\frac{\hat{r}}{r^2}$$

$$\therefore\ \boxed{\vec{B}^{(1)} = -\vec{\nabla}\underbrace{\left(\frac{\hat{r}\cdot\vec{m}}{r^2}\right)}_{\text{pseudo potential}} \quad \text{"Field of dipole}}$$

Compare to electric dipole: $\vec{E} = -\vec{\nabla}\left(\hat{r}\cdot\frac{\vec{p}}{r^2}\right)$

Two important cases

(i) Planar current loop:



area swept out $d\vec{a} = \frac{1}{2}\vec{x} \times d\vec{l}$

$$\vec{m} = \frac{1}{2c}\int d^3x \;\; \vec{x} \times \vec{J}(\vec{x}) = \frac{I}{2c}\oint \vec{x} \times d\vec{l}$$

$$= \frac{I}{c}\oint da\, \hat{n} = \boxed{\frac{I}{c}\,\text{Area}\;\hat{n} = \vec{m}}$$

(ii) Point charge in orbit: $\vec{J}(\vec{x}) = q\vec{v}_p\, \delta^{(3)}(\vec{x} - \vec{x}_p(t))$

$$\vec{m} = \frac{1}{2c}\int d^3x \;\; \vec{x} \times \vec{J}(\vec{x}) = q\,\frac{\vec{x}_p \times \vec{v}_p}{2c}$$

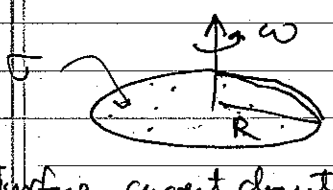
$$\Rightarrow \boxed{\vec{m} = \frac{q}{2mc}\,\vec{L}_{\text{Particle}}}$$

$\vec{L}$ = orbital angular momentum

$\frac{q}{2mc} \equiv$ gyromagnetic ratio

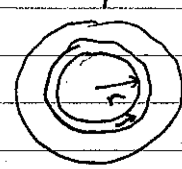
Not exact since current not steady state. Approximate if $v \ll c$

Example: Rotating disk of surface charge



Surface current density
$K = \sigma\omega r$

Break up into rings



$$dI = K(r)\,dr = \sigma\omega r\,dr$$

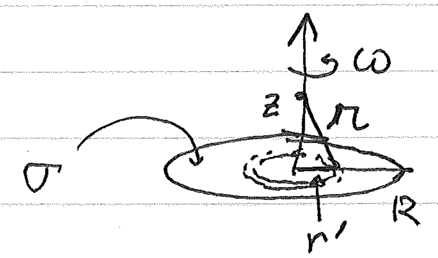
$$\Rightarrow d\vec{m} = \frac{dI}{c}\pi r^2 = \frac{1}{c}\sigma\omega\pi r^3 dr$$

$$\Rightarrow \vec{m} = \int_0^R dm(r)\;\vec{e}_z = \frac{\pi\sigma\omega R^4}{4c}$$

Far away from disk $\quad \vec{B} = \vec{\nabla}\left(\hat{r}\cdot\frac{\vec{m}}{r^2}\right) \quad r \gg R$

On z-axis $\quad \vec{B}(z) \simeq \frac{2m}{z^3}\vec{e}_z = \frac{\pi\sigma\omega R^4}{2cz^3}\,\vec{e}_z \quad$ (Compare to limit of exact expression)

Magnetic field of a rotating disk of charge



Surface current density

$$K = \sigma v = \sigma \omega r$$

$$dB_z(z) = \frac{2\pi r'^2}{(z^2+r'^2)^{3/2}}\, dI$$ ← current in differential ring

$$dI = K\, dr'$$

$$\Rightarrow dB_z(z) = \frac{2\pi\sigma\omega}{c}\, \frac{r'^3\, dr'}{(z^2+r'^2)^{3/2}}$$

$$\therefore B_z(z) = \frac{2\pi\sigma\omega}{c}\int_0^R \frac{r'^3\, dr'}{(z^2+r'^2)^{3/2}} = \frac{2\pi\sigma\omega}{c}\left(\frac{r'^2+2z^2}{\sqrt{r'^2+z^2}}\right)_0^R$$

$$\Rightarrow B(z) = \frac{2\pi\sigma\omega}{c}\left(-2z + \frac{R^2+2z^2}{\sqrt{R^2+z^2}}\right)$$

Limits

$z \ll R$ $\quad B(z) \to \dfrac{2\pi\sigma\omega R}{c} = \dfrac{2\pi K(R)}{c}$

$z \gg R$ $\quad B(z) = \dfrac{2\pi\sigma\omega}{c}\left(-2z + \left(\dfrac{R^2}{z}+2z\right)\left(1+\dfrac{R^2}{z^2}\right)^{-1/2}\right)$

$$\cong \frac{2\pi\sigma\omega}{c}\left(-2z + \left(\frac{R^2}{z}+2z\right)\left(1 - \frac{R^2}{2z^2} + \frac{3}{8}\frac{R^4}{z^4}\right)\right)$$

$$= \frac{2\pi\sigma\omega}{c}\left(-2z + \frac{R^2}{z} + 2z - \frac{R^4}{2z^3} - \frac{R^2}{z} + \frac{3}{4}\frac{R^4}{z^3}\right)$$

$$\Rightarrow \boxed{B(z) = \frac{2m}{z^3}}, \quad \text{magnetic dipole: } m = \pi\sigma\omega R^4/4c$$

Forces on currents in external fields

$$\vec{F} = \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}') \times \vec{B}(\vec{x}') \Rightarrow F_i = \frac{1}{c}\epsilon_{ijk}\int d^3x'\, J_j(\vec{x}')\, B_k(\vec{x}')$$

Distribution localized at origin $\Rightarrow$ Expand $B_k(\vec{x}')$

$$B_k(\vec{x}') \cong B_k(0) + \vec{x}'\cdot\vec{\nabla}B_k\Big|_{x=0} + \cdots$$

$$\Rightarrow F_i = \frac{1}{c}\epsilon_{ijk}\underbrace{\left(\int d^3x'\, J_j(\vec{x}')\right)}_{=\,0} B_k(0) + \epsilon_{ijk}\underbrace{\left[\frac{1}{c}\int d^3x'\, J_j(\vec{x}')\,x_l'\right]}_{=\,\epsilon_{ljp}\,m_p}\partial_l B_k\Big|_0$$

Aside

$$\epsilon_{ijk}\,\epsilon_{ljp} = \epsilon_{ikj}\,\epsilon_{jlp} = \delta_{il}\,\delta_{kp} - \delta_{ip}\,\delta_{kl}$$

$$\Rightarrow F_i = m_k\,\partial_i B_k - m_i\,\partial_k B_k$$

$$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) - \vec{m}\,(\vec{\nabla}\cdot\vec{B}) \Rightarrow \boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})}$$

Potential energy: $\boxed{U = -\vec{m}\cdot\vec{B}}$

- Constant field: Torque $\vec{N} = \vec{m}\times\vec{B}$
- Gradient field: Trap