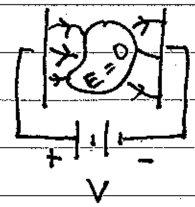
Faraday's Law and Quasi-statics - Lecture #9

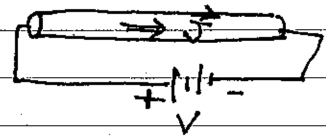
Beyond Statics

- Electro-statics



Electric field is zero inside the conductor

- Magnetostatics



Current flowing in conductor ⇒ Must be a force on charges

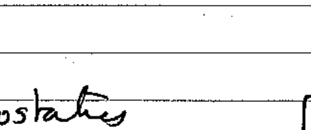
Since work is done on charges, must be due to $\vec{E}$-field

Phenomenological relation $\quad \vec{J} = \sigma_c \vec{E} \quad$ (Ohm's "Law")

σ_c = Conductivity $\quad$ (Generally tensor $\vec{J} = \overleftrightarrow{\sigma}_c \cdot \vec{E}$)

Drude model: Free electron gas, collisions with defects

Collisions w/ freq ν_e



$\overrightarrow{\langle \vec{v} \rangle}$ = drift velocity

Macroscopic current

$\vec{J} = -e n_e \langle \vec{v} \rangle$

$\Uparrow$ density of electrons

Equation of motion: $\quad m_e \dfrac{d\langle \vec{v} \rangle}{dt} = -e\vec{E} - \underbrace{(m_e \nu_e) \langle \vec{v} \rangle}_{\text{Drag force}}$

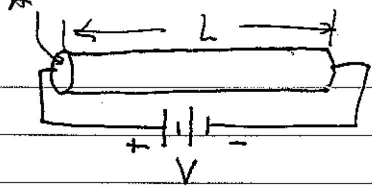
$$\Rightarrow \boxed{\frac{d\vec{J}}{dt} + \nu_e \vec{J} = \frac{\omega_p^2}{4\pi} \vec{E}} \qquad \omega_p^2 = \frac{4\pi n_e e^2}{m_e} \quad \left(\text{Plasma frequency}\right)$$

Generalized Ohm's Law

D.C. / Steady state limit $\quad T \gg \frac{1}{\nu_e}$

$$\Rightarrow \boxed{\vec{J} = \frac{\omega_p^2}{4\pi \nu_e} \vec{E} = \sigma_c \vec{E}\,; \quad \sigma_c = \frac{\omega_p^2}{4\pi \nu_e}}$$

<u>Simple geometry:</u>



Ignoring fringe fields $E = \frac{V}{L}$, $J = \frac{I}{A}$

$$J = \sigma_c E \Rightarrow \frac{I}{A} = \sigma_c \frac{V}{L} \Rightarrow V = \left(\frac{L}{A\sigma_c}\right) I$$

$$\Rightarrow \boxed{V = IR} \qquad \text{Resistance } R = \frac{L}{A}\rho_c \leftarrow \frac{1}{\sigma_c} : \text{resistivity}$$

Because of resistance, flowing charges lose energy

<u>Ohmic heating:</u> Rate at which work is done on charges

$$\vec{F}_i = q_i \vec{E}_i + q_i \frac{\vec{v}_i}{c} \times \vec{B}_i \Rightarrow \frac{dW}{dt} = \sum_i \vec{v}_i \cdot \vec{F}_i$$

$$\Rightarrow \frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}_i = \int d^3x \, \vec{J}(\vec{x}) \cdot \vec{E}(\vec{x})$$

$$\left(\vec{J}(x) = \sum_i q \vec{v}_i \, \delta(\vec{x} - \vec{x}_i)\right)$$

$$\left(= IV = I^2 R \text{ for simple geometry}\right)$$

$$\Rightarrow \boxed{\vec{J} \cdot \vec{E} = \text{rate of energy dissipation / volume}}$$

<u>EMF:</u> If energy is being dissipated, then the battery is doing work

$$\oint_c \vec{F} \cdot d\vec{l} = \text{Work done to push charge around circuit}$$

$$\vec{F} = \vec{F}_{\text{battery}} + \vec{F}_{\text{electrostatic}}$$

Define $\boxed{\mathcal{E} = \frac{1}{q}\oint \vec{F} \cdot d\vec{l} : \underset{= \text{Voltage}}{\text{Work/charge}} \quad \left(\begin{matrix}\text{Historic name} & \text{EMF} \\ \text{"Electromotive Force"}\end{matrix}\right)}$

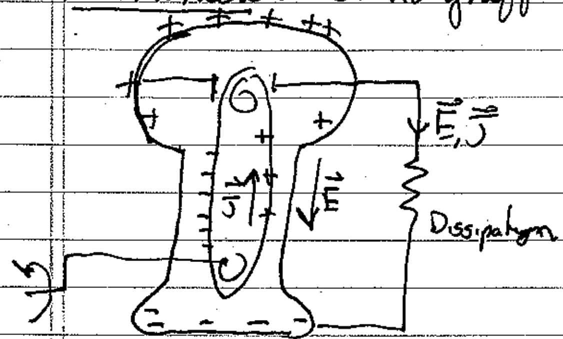
$$\mathcal{E} = \frac{1}{q}\oint \vec{F}_{\text{battery}} \, d\vec{l} + \underbrace{\frac{1}{q}\oint \vec{F}_{\text{electrostatic}} \cdot d\vec{l}}_{0}$$

Since $\oint_c \vec{E} \cdot d\vec{l} = 0$

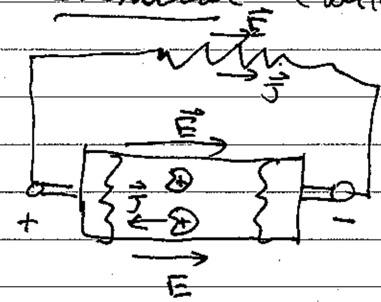
Electrostatic fields cannot provide EMF (conservative field)

Examples of EMF

- Mechanical (Vandegraff)

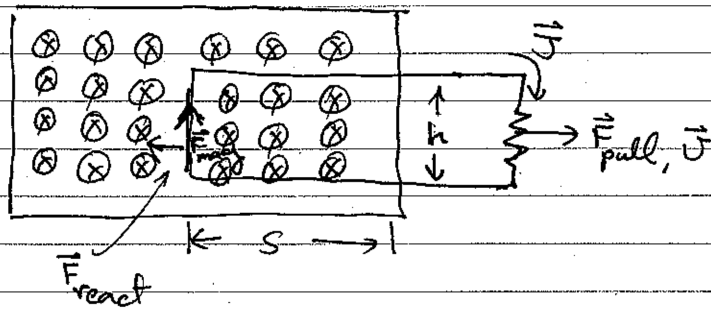


- Chemical (battery)



- "Motional" EMF : Lorentz Force

Uniform $\vec{B}$-field



$|\vec{F}_{mag}| = q\frac{v}{c}B$

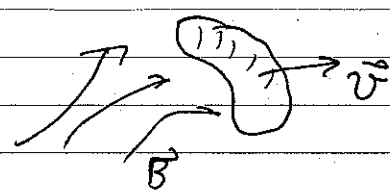
(Lorentz force on electrons inside the wire as they are dragged through $\vec{B}$)

$$\mathcal{E} = \oint \frac{\vec{F}}{q}\cdot d\vec{\ell} = \frac{v}{c}Bh = \frac{1}{c}\left(-\frac{ds}{dt}\right)hB = -\frac{1}{c}\frac{d\Phi_B}{dt}$$

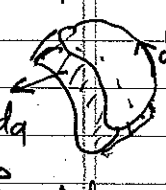
$$\boxed{\mathcal{E} = -\frac{1}{c}\frac{d\Phi_B}{dt} \qquad \Phi_B = \int_S \vec{B}\cdot d\vec{a} = (sh)\,B}$$

General geometrical feature:

Loop moving through $\vec{B}$-field with velocity $\vec{v}$:



Loop sweeps out volume:



$d\vec{a} = \hat{n}\,da = d\vec{\ell}\times\vec{v}\,dt$

$$\oint_S \vec{B}\cdot d\vec{a} = 0 \Rightarrow \Phi_B(t+dt) - \Phi_B(t) = -\int_{\text{sides}} \vec{B}\cdot d\vec{a}$$

$$\Rightarrow d\Phi_B = -dt\oint_C \vec{B}\cdot(d\vec{\ell}\times\vec{v}) = +\oint(\vec{B}\times\vec{v})\cdot d\vec{\ell}\;dt$$

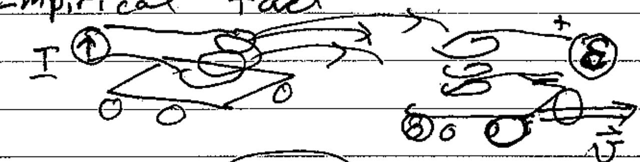
~~[illegible]~~ $\Rightarrow \; d\Phi_B = -\oint_C (\vec{v}\times\vec{B})\cdot d\vec{l}\, dt$

$$\Rightarrow \frac{d\Phi_B}{dt} = -c\oint \frac{\vec{F}}{q}\cdot d\vec{l} \;\Rightarrow\; \boxed{\mathcal{E}_{motional} = -\frac{1}{c}\frac{d\Phi}{dt}},$$

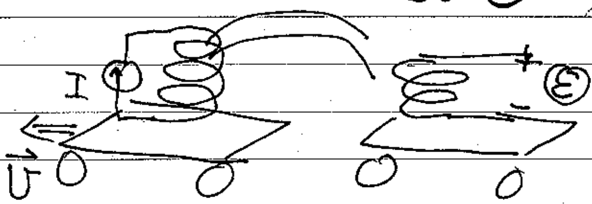
No new physics, just Lorentz force

Faraday's Universal Law of Induction

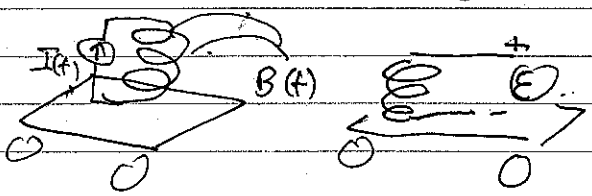
Empirical fact

(1) 

Move coil through non-uniform $\vec{B}$-field

(2) 

Move source of $\vec{B}$-field, keep coil fixed

(3) 

Keep coil + B-field fixed, change B(t) by changing I(t)

All give the same answer $\boxed{\mathcal{E} = -\frac{1}{c}\frac{d\Phi_B}{dt}}$

Lenz's Law

Modern understanding : Lorentz invariance (Einstein)

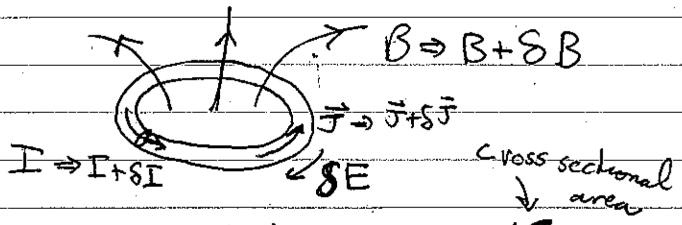
In rest frame of the coil the force that moves charge must be an electric field

$\Rightarrow$ In rest frame of circuit $\oint \frac{\vec{F}}{q}\cdot d\vec{l} = \boxed{\oint_C \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}}$

New $\vec{E}$-field : $\boxed{\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}$.

Energy in the Magnetic Field

Work done to establish a current distribution



Current loop $I = J\, dS$

As the current increases the flux of B increases. There is a "back EMF" to oppose this change. We must do work to balance this force.

Rate at which work is done: $\underline{I \Rightarrow I + \delta I}$

$$\frac{d}{dt}(\delta W) = -\int \vec{J}\cdot \delta\vec{E}\, \underset{\int dS \oint d\vec{l}}{\underset{\shortparallel}{d^3x}} = -\int J dS \oint \delta\vec{E}\cdot d\vec{l} = \frac{1}{c}\int J dS \int \frac{\delta\vec{B}}{\delta t}\cdot d\vec{a}$$

$$\Rightarrow \delta W = \frac{1}{c}\int dS\, J \int \delta\vec{B}\cdot d\vec{a} = \frac{1}{c}\int dS\, J\int \vec{\nabla}\times(\delta\vec{A})\cdot d\vec{a}$$

$$= \frac{1}{c}\int dS\, J \oint \delta\vec{A}\cdot d\vec{l} = \frac{1}{c}\int \vec{J}\cdot\delta\vec{A}\, d^3x$$

$$\therefore \boxed{W = \frac{1}{2c}\int \vec{J}\cdot\vec{A}\, d^3x = U_B \ \left(\begin{matrix}\text{Potential energy}\\ \text{stored in } \vec{B}\end{matrix}\right)}$$

Express in terms of the field:

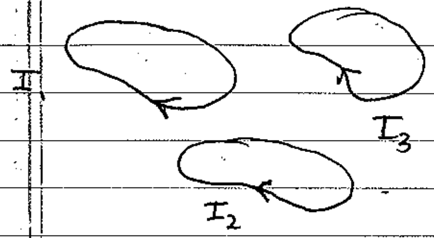
$$U = \frac{1}{2c}\int \left(\frac{c}{4\pi}\vec{\nabla}\times\vec{B}\right)\cdot\vec{A}\, d^3x$$

Integrate by parts

$$\boxed{U_B = \frac{1}{8\pi}\int \vec{B}\cdot\vec{B}\, d^3x}$$

L#9.6

<u>Collection of loops</u>



$$U_B = \frac{1}{2c}\int d^3x\, \vec{J}\cdot\vec{A}$$

$$= \sum_i \frac{1}{2c}\oint d\vec{l}_i\, I_i \cdot \vec{A}(\vec{x}_i)$$

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{d^3x'\,\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{c}\sum_j \oint \frac{d\vec{l}_j\, I_j}{|\vec{x}-\vec{x}_j|}$$

$$\Rightarrow U_B = \frac{1}{2}\sum_{ij}\left(\frac{1}{c^2}\oint\!\!\oint \frac{d\vec{l}_i\cdot d\vec{l}_j}{|\vec{x}_i-\vec{x}_j|}\right) I_i I_j$$

$$\Rightarrow \boxed{U_B = \frac{1}{2}\sum_{i,j} L_{ij}\, I_i I_j\,, \quad L_{ij} = \frac{1}{c^2}\oint\!\!\oint \frac{d\vec{l}_i\cdot d\vec{l}_j}{|\vec{x}_i-\vec{x}_j|}}$$

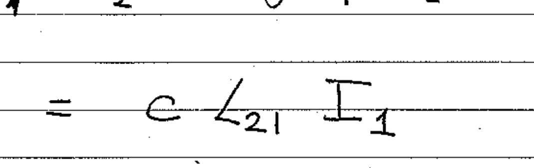
L_{ij} = coefficient of mutual inductance

$= L_{ji}$ (reciprocity), $L_{ii} \equiv L$ = self-inductance

Single loop $U_B = \frac{1}{2} L I^2$, L <u>geometrical</u>

Units in Gaussian $[L] = [\frac{1}{c^2}]\,\text{cm} \equiv$ Oerstead (S.I.: Henry)

<u>Calculating inductance</u>
(Look at changing flux)



$$\Phi_{2,1} = \int_{S_2} \vec{B}_1\cdot d\vec{a}_2 = \int_S (\vec{\nabla}\times\vec{A})_1\cdot d\vec{a}_2 = \oint \vec{A}_1\cdot d\vec{l}_2$$

$$= \frac{1}{c}\left(\oint\!\!\oint \frac{d\vec{l}_1\cdot d\vec{l}_2}{|\vec{x}_2-\vec{x}_1|}\right) I_1 = c\, L_{21}\, I_1$$

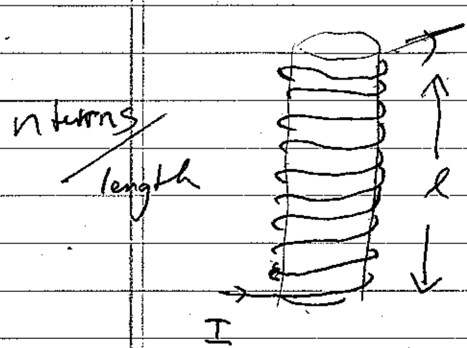
$$\Rightarrow \boxed{L_{2,1} = \frac{\Phi_{21}}{c I_1}\,, \quad L = \frac{\Phi}{cI}} \qquad \mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$$

$$\Rightarrow \boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

Inductors oppose changing current

L#9.7

Example: Solenoid



Assume current I

$\Rightarrow$ Uniform $B = \frac{4\pi}{c} n I$

Total flux:

$$\Phi = N_{turns} BA = n \ell B A$$

$$\Phi = \frac{4\pi}{c} n^2 \ell A I$$

$$\Rightarrow \boxed{L = \frac{\Phi}{cI} = \frac{4\pi n^2 \ell A}{c^2}}$$