Lecture #10 : Maxwell's Eqns and Conservation Laws

Summary of field eqns up to this point

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$$

But we cheated. Ampère's Law here is for steady currents, but we've used it in Faraday's Law. When is it valid? (See below)

Mawell generalizes Ampère's Law

Add an additional piece: $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \vec{J}_d$

← Mawell's Displacement current

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = \frac{4\pi}{c}(\vec{\nabla}\cdot\vec{J}) + \vec{\nabla}\cdot\vec{J}_d$$

But $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$: Conservation of charge

$$\Rightarrow \vec{\nabla}\cdot\vec{J}_d = \frac{4\pi}{c}\frac{\partial\rho}{\partial t} = \frac{4\pi}{c}\frac{\partial}{\partial t}\left(\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}\right) = \vec{\nabla}\cdot\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\cdot\left(\vec{J}_d - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) = 0$$

$$\therefore \boxed{\vec{J}_d = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}} + \cancel{\vec{Q}} \to 0 \quad \text{where } \vec{\nabla}\cdot\vec{Q} = 0$$

Empirical fact

Maxwell's Displacement current

General Form of Ampère's Law

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}}$$

Changing $\vec{E}$ generates $\vec{B}$

#10.1

Why was the displacement current missed empirically?

Compare scales: Suppose T is characteristic time scale ~~scale~~ of change, L is distance

Faraday: $\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{E}{L} = \frac{1}{cT} B$

Ampere: $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{B}{L} \sim \frac{J}{c} + \left(\frac{L}{cT}\right)^2 \frac{B}{L}$

Conduction current will mask effects of displacement current if $\frac{L}{cT} \ll 1 \Rightarrow \boxed{\frac{1}{T} \ll \frac{c}{L}}$

Fact: · Magnetic forces much weaker than electric unless relativistic $v \sim c$

<u>Quasi-statics</u> Ignore displacement current

Valid if $cT \gg L$

However, even in non-relativistic sitation, if

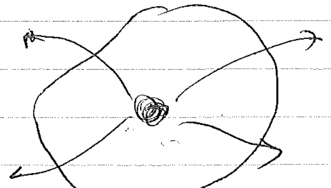
$L \gg cT$ (i.e. far away from circuit)

effects of displacement current are important

$\Rightarrow$ **Radiation!**

Conservation Laws

General form: Q = conserved quantity



Define ρ_Q = density of Q $\left[\frac{Q}{L^3}\right]$

$\vec{F}_Q$ = Flux density of Q $\left[\frac{Q/T}{L^2}\right]$

R_Q = Rate of creation of Q/vol $\left[\frac{Q}{T L^3}\right]$

Conservation $\equiv$ detailed balance

$$\frac{dQ}{dt} = -(\text{Rate of flow of } Q \text{ out of volume}) + \left(\begin{array}{c}\text{Rate of creation} \\ \text{of } Q \\ \text{inside volume}\end{array}\right)$$

$$\frac{d}{dt}\int_V \rho_Q \, d^3x = -\oint_S \vec{F}_Q \cdot d\vec{a} + \int_V R_c \, d^3x$$

Differential form
$$\boxed{\frac{\partial \rho_Q}{\partial t} = -\vec{\nabla}\cdot\vec{F}_Q + R_c}$$

e.g.: charge $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$ (No creation rate)

Energy in E/M field

$$U = \frac{1}{8\pi}\int (\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})\, d^3x \, , \qquad u = \frac{(\vec{E}^2 + \vec{B}^2)}{8\pi} \text{ energy density}$$

$$\frac{\partial}{\partial t} u = \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right) = \frac{1}{4\pi}\left(\vec{E}\cdot(c\vec{\nabla}\times\vec{B} - 4\pi\vec{J}) + \vec{B}\cdot(-c\vec{\nabla}\times\vec{E})\right)$$

$$= \frac{c}{4\pi}\left(\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})\right) - \vec{J}\cdot\vec{E}$$

$$= -\frac{c}{4\pi}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \vec{J}\cdot\vec{E}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E} \, , \qquad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \text{ Poynting vector}}$$

Poynting's theorem

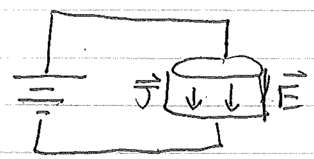
Poynting vector: $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$: energy flux density

$\int_{\text{surface}} \vec{S}\cdot d\vec{a}$ = rate of energy flow through surface

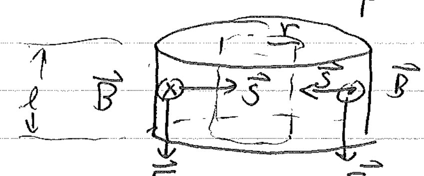
$-\vec{J}\cdot\vec{E}$ = Rate of creation of energy density in field
= −(Ohmic heating rate / volume)

- Examples of Energy Conservation

Ohmic dissipation: Resistor



Close up



$$B = \frac{2\, I_{enc}(r)}{cr} = \frac{2\pi}{c} J r$$

(ignore displacement current $r \ll cT$)

$$\Rightarrow \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{r}{2} J E\, \vec{e}_\rho$$

Flux through side of ~~side~~ cylinder: $\int \underset{\text{outward}}{\vec{S}\cdot d\vec{a}} = -(2\pi r\, \ell)\frac{r}{2} J E =$

$$\Rightarrow \int \vec{S}\cdot d\vec{a} = -(\pi r^2 \ell)\, J E = -IV$$ Ohmic dissipation

Uniqueness of $\vec{S}$? Physics $\vec{\nabla}\cdot\vec{S}$

In statics more natural choice $\vec{S} = \vec{J}\phi$ $\left(\frac{IV}{\text{Area}}\right)$

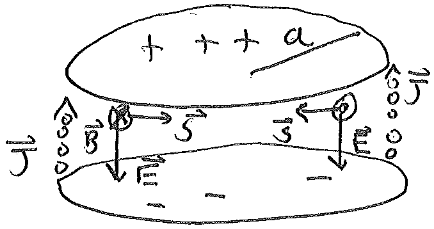
Check: $\vec{\nabla}\cdot(\phi\vec{J}) = \vec{J}\cdot(\vec{\nabla}\phi) + \phi\,\cancel{(\vec{\nabla}\cdot\vec{J})}^{\,0} = -\vec{J}\cdot\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E} \quad ✓$$

Not equivalent in dynamics, including radiation

L#10.4

Example: Create E+M energy : Charge a capacitor by moving current against field



Suppose current is moved only along the edges of the plate
$\Rightarrow$ only displacement current inside

Ampere $\oint \vec{B}\cdot d\vec{l} = \frac{1}{c}\int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a} \Rightarrow B(r) = \frac{1}{2\pi r c}\int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$

Neglect fringing fields $\Rightarrow$ E uniform inside $\Rightarrow \vec{B} = -\frac{r}{2c}\frac{\partial E}{\partial t}\hat{\phi}$

$$\Rightarrow \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{r}{8\pi}|\vec{E}|\left|\frac{\partial \vec{E}}{\partial t}\right|\hat{e}_\rho = -\frac{r}{16\pi}\frac{\partial}{\partial t}|\vec{E}|^2\,\hat{e}_\rho$$

<u>Just inside the cylinder</u>

$$\int \vec{S}\cdot d\vec{a} = -(2\pi a \underset{\uparrow}{d})\frac{a}{16\pi}\frac{\partial}{\partial t}|\vec{E}|^2 = -(\pi a^2 d)\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi}\right)$$

distance between plate

What happened to $\vec{B}$? Quasi-static, ignore back EMF

$$\Rightarrow \vec{\nabla}\times\vec{E} \cong 0 \quad \Rightarrow \quad \vec{E}(t) \approx -4\pi\frac{Q(t)}{A}\vec{e}_z$$

Energy created at edge a flows into the capacitor

$$-\int \vec{J}\cdot\vec{E}\,d^3x = \cancel{\quad} \; -\int \vec{J}\cdot d\vec{a}\,dz\,\vec{E} = \left(\frac{dQ}{dt}\right)|\vec{E}|d$$

$$= \left(\frac{A}{4\pi}\frac{\partial|\vec{E}|}{\partial t}\right)|\vec{E}|\,d = (Ad)\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi} \quad \checkmark$$

This is an approximation since energy created at edge immediately appears inside capacitor: <u>Neglect radiation</u>

Valid if $T \gg L/c$