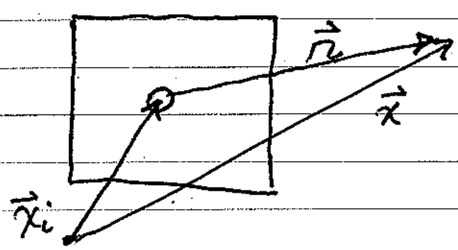
Lecture 14: Dielectrics and Magnetic Materials

- Macroscopic Field Eqns.

Prescribed sources in surrounding media
- Macroscopic observations do not depend on detailed behavior of fields on microscopic distances

"Coarse grain": Average over ΔV $\begin{cases} \gg \text{atom scale } \mathring{A} \\ \ll \text{macroscopic } (\mu m) \end{cases}$



Coarse graining function $f(\vec{x})$

$$\Rightarrow \rho_{macro}(\vec{x},t) = \int d^3x' \; f(\vec{x}-\vec{x}') \, \rho_{micro}(\vec{x}',t)$$

$$\vec{E}_{macro}(\vec{x},t) = \int d^3x' \; f(\vec{x}-\vec{x}') \; \vec{E}_{micro}(\vec{x}',t)$$

Can Show: $\vec{\nabla}\cdot\vec{E}_{macro} = 4\pi\rho_{macro}$ etc.

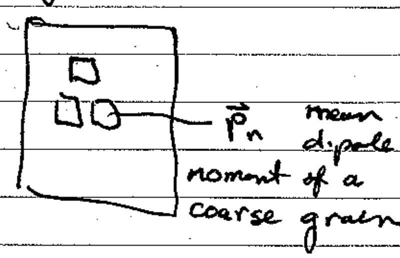
derivative w.r.t. coarse grained position

Can divide $\rho_{macro} = \rho_{free} + \rho_{bound}$ bound in molecules

Lowest order multipole: Dipole density $\vec{P}(\vec{x},t)$

$$\vec{P}(\vec{x},t) = \sum_n \vec{p}_n \, f(\vec{x}-\vec{x}_n)$$

"Polarization (density)" = Dipole/Volume



$\Rightarrow$ Macroscopic "bound charge density"

Magnetization $\vec{M} = \sum_n \vec{m}_n f(\vec{x}-\vec{x}_n)$: $\dfrac{\text{Aver magnetic dipole moment}}{\text{Volume}}$

Averaging ⇒ outside a "grain" view as a collection of point multipoles.

Neutral Molecule ⇒ lowest non-vanishing moment is dipole.

$$\phi_{dipole} = \int d\vec{p} \cdot \frac{\hat{r}}{r^2} = \int d^3x' \, \vec{P}(\vec{x}') \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = +\int d^3x' \, \vec{P}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|}$$

$$= \oint_S \frac{\vec{P} \cdot \hat{n} \, da'}{|\vec{x} - \vec{x}'|} + \int \frac{-\vec{\nabla} \cdot \vec{P}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \quad \rho_{bound}$$

Bound charge density: $\rho_{bound} = -\vec{\nabla} \cdot \vec{P}$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_{total} = 4\pi \rho_{bound} + 4\pi \rho_{free}$$

$$\Rightarrow \vec{\nabla} \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho_{free}$$

$$\underbrace{\vec{E} + 4\pi \vec{P}}_{\equiv \vec{D}}$$

$\vec{D}$ Displacement field

$$\therefore \boxed{\vec{\nabla} \cdot \vec{D} = 4\pi \rho_{free}}$$

$\boxed{\Delta D_\perp = 4\pi \sigma_{free}}$ Boundary condition

Macroscopic current density

$$\vec{J}_{macro} = \vec{J}_{free} + \vec{J}_{bound}$$

$$\underset{\substack{\uparrow \\ \text{magnetization}}}{\vec{J}_{bound} = \vec{J}_{mag}} + \underset{\substack{\nwarrow \\ \text{time varying dipole}}}{\vec{J}_{pol}}$$

- Magnetization current

$$\vec{A}_{mag} = \int d^3x' \; \vec{M}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \underbrace{\oint_S \frac{\vec{M}\times\hat{n}\,da}{|\vec{x}-\vec{x}'|}}_{\text{surface contribution}} + \int \frac{\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

$$= \frac{1}{c}\int \frac{\vec{J}_{mag}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

$$\boxed{\vec{J}_{mag}(\vec{x}') = c\,\vec{\nabla}'\times\vec{M}(\vec{x}')}$$

- Polarization current density

$$\frac{\partial \rho_{bound}}{\partial t} = -\vec{\nabla}\cdot\vec{J}_{pol} = \frac{\partial}{\partial t}\left(-\vec{\nabla}\cdot\vec{P}\right)$$

$$\Rightarrow \boxed{\vec{J}_{pol} = \frac{\partial \vec{P}}{\partial t}}$$

∴ Macroscopic Ampère - Maxwell

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\left(\vec{J}_{free} + c\vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Define $\vec{H} = \vec{B} - 4\pi\vec{M}$: Auxilary magnetic field

$$\Rightarrow \vec{\nabla}\times(\vec{B}-4\pi\vec{M}) = \frac{4\pi}{c}\vec{J}_{free} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}+4\pi\vec{P})$$

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_{free} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}}$$

Displacement current $\frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}$

In vacuum; $\frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$ (oscillating either)

<u>Macroscopic Maxwell's Eqns:</u>

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_{free} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{H} = \vec{B} - 4\pi\vec{M}$$

Boundary conditions

$$\boxed{\begin{array}{ll} \Delta D_\perp = 4\pi\sigma_{free} & \Delta B_\perp = 0 \\ \Delta E_\parallel = 0 & \Delta H_\parallel = \frac{4\pi}{c}K_{free} \end{array}}$$

L#14.5

Constitutive relations

In most materials, no permanent $\vec{P}$ or $\vec{M}$, but induced by external field. For weak fields, these are linearly proportional to the external fields. We define "susceptibility" for a given frequency and spatial mode of the applied field

$$\underset{\sim}{P}(\omega,\vec{k}) = \overleftrightarrow{\chi}_e(\omega,\vec{k}) \cdot \underset{\sim}{E}(\omega,\vec{k}) \quad \left(\begin{array}{c}\text{Tensor}\\ \underline{\text{response}}\end{array}\right)$$

Typically, uniform dielectric $\underset{\sim}{P}(\vec{x},\omega) = \overleftrightarrow{\underset{\sim}{\chi}}_e(\omega) \cdot \underset{\sim}{\vec{E}}(\vec{x},\omega)$

$$\Rightarrow \vec{P}(\vec{x},t) = \int_0^t dt' \, \overleftrightarrow{\chi}_e(t-t') \cdot \vec{E}(\vec{x},t')$$

Similarly, define

$$\underset{\sim}{\vec{M}}(\vec{x},\omega) = \overleftrightarrow{\chi}_m(\omega) \cdot \underset{\sim}{\vec{H}}(\vec{x},\omega)$$

magnetic susceptibility tensor

$$\Rightarrow \vec{M}(\vec{x},t) = \int_0^t dt' \, \overleftrightarrow{\chi}_m(t-t') \cdot \vec{H}(\vec{x},t')$$

The width of the functions $\chi_e(t-t')$, $\chi_m(t-t')$ determine the "response time" of the materials.

For a range of frequencies where response time << rate of change of fields ⇒ frequency independent ⇒ "non-dispersive"

$$\Rightarrow \vec{P}(\vec{x},t) \simeq \overleftrightarrow{\chi}_e \cdot \vec{E}(\vec{x},t)$$

"instanteous response"

$$\vec{M}(\vec{x},t) \simeq \overleftrightarrow{\chi}_m \cdot \vec{H}(\vec{x},t)$$

$$\Rightarrow \vec{D} = \vec{E} + 4\pi\vec{P} = \underbrace{(\overleftrightarrow{\mathbb{1}} + 4\pi\overleftrightarrow{\chi}_e)}_{\equiv\ \overleftrightarrow{\epsilon}} \cdot \vec{E}$$

$\equiv \overleftrightarrow{\epsilon}$: Dielectric permittivity tensor

$$\vec{B} = \vec{H} + 4\pi\vec{M} = \underbrace{(\overleftrightarrow{\mathbb{1}} + 4\pi\overleftrightarrow{\chi}_m)}_{\equiv\ \overleftrightarrow{\mu}} \cdot \vec{H}$$

$\equiv \overleftrightarrow{\mu}$: Magnetic permeability tensor

Susceptibility from microscopic response

Single molecule/atom polarizability $\overleftrightarrow{\alpha}$

$$\Rightarrow \vec{p} = \overleftrightarrow{\alpha} \cdot \vec{E} \quad \text{(dipole induced)}$$

Dilute gas, $N \frac{\text{molecules}}{\text{Volume}} \Rightarrow \overleftrightarrow{\chi} = N\overleftrightarrow{\alpha}$

More generally, $\vec{P} = N\overleftrightarrow{\alpha}(\vec{E}_{external} + \vec{E}_{\text{local fields}})$

Due surrounding medium

e.g. $$\chi_e = \frac{N\alpha}{1 - \frac{4\pi}{3}N\alpha}$$

L # 14.8

For isotropic, uniform, nondispersive response

$$\vec{D} = \epsilon \vec{E}, \qquad \vec{B} = \mu \vec{H}$$

$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = 4\pi \rho/\epsilon & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\mu\vec{J} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t} \end{array}}$$

Notes on units:

In c.g.s. units $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ all have same units

$\Rightarrow$ $\vec{P}$ and $\vec{M}$ have same units as $\vec{E}, \vec{B}$

Check $[\vec{P}] = \frac{\text{dipole}}{\text{Volume}} = \frac{\text{Charge}\cdot\text{Distance}}{(\text{Distance})^3} = \frac{\text{Charge}}{(\text{Distance})^2}$ ✓

$\Rightarrow$ $\overleftrightarrow{\chi}$, $\overleftrightarrow{\epsilon}$, $\overleftrightarrow{\mu}$ dimensionless

Contrast with SI units $\vec{P} = \epsilon_0 \overleftrightarrow{\chi}\cdot\vec{E}$

Permittivity $\overleftrightarrow{\epsilon} = \epsilon_0 \underbrace{(\mathbb{1} + \overleftrightarrow{\chi})}$ ← dielectric constant tensor

↑ permittivity of "free space"

$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\epsilon & \vec{\nabla}\cdot\vec{B} = 0 \qquad (\text{SI}) \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial \vec{E}}{\partial t} \end{array}}$$

Energy transport in fields in the presence of ponderable materials

$-\vec{J}\cdot\vec{E}$ = rate at ~~the~~ which fields do work on free source

$$= -\vec{E}\cdot\left(\frac{c}{4\pi}\vec{\nabla}\times\vec{H} - \frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}\right)$$

Aside: Consider $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \partial_i \epsilon_{ijk} E_j H_k$

$$= \epsilon_{ijk}\left(H_k \partial_i E_j + E_j \partial_i H_k\right)$$

$$= \vec{H}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{= -\frac{1}{c}\frac{\partial B}{\partial t}} - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \boxed{-\vec{J}\cdot\vec{E} = \frac{c}{4\pi}\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)}$$

$\Rightarrow$ Poyting vector in medium $\boxed{\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{H})}$

The energy density is not simple in general, and medium response time transfer energy between field and matter at finite rate.

For ~~instanteoneous~~ instantaneous linear response, $\vec{D} = \overleftrightarrow{\epsilon}\cdot\vec{E}$
$\vec{B} = \overleftrightarrow{\mu}\cdot\vec{H}$

$$\Rightarrow \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) \qquad \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H})$$

$\Rightarrow$ Energy density $\boxed{u = \frac{\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}}{8\pi}}$