Lecture #16 Wave propagation in dispersive media

Simple models of dispersion

The constitutive relations we've assumed are naive

$$\vec{P}(\vec{x},t) = \chi_e \vec{E}(\vec{x},t), \quad \vec{M}(\vec{x},t) = \chi_m \vec{H}(\vec{x},t), \quad \vec{J}(\vec{x},t) = \sigma \vec{E}(\vec{x},t)$$

Assumes that the response of the medium to the field is instaneous

More realistic: χ_e, σ, χ_m represent steady state response coefficients to a drive at freq ω

$\Rightarrow$ Local relation in the frequency domain

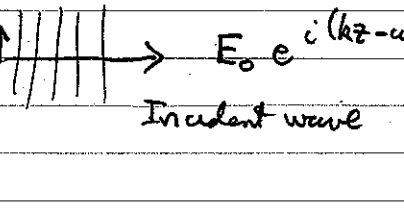
$$\underset{\sim}{\vec{P}}(\vec{x},\omega) = \tilde{\chi}_e(\omega)\, \underset{\sim}{\vec{E}}(\vec{x},\omega) \quad \text{in Fourier space}$$

$$\Rightarrow \vec{P}(\vec{x},t) = \int \frac{d\omega}{2\pi} \underset{\sim}{\vec{P}}(\vec{x},\omega)\, e^{-i\omega t} = \int dt' \chi_e(t-t')\, \vec{E}(\vec{x},t')$$

($\chi_e(t-t')$: response function)

Lorentz Model of dielectrics

Bound charges in molecules modeled by charges on spring



(charge on spring, ω_0 = natural resonance, Γ = damping coef.)

Eqn of motion:

$$m\ddot{x} = -m\omega_0^2 x - m\Gamma\dot{x} - eE(z=0,t)$$

$\Rightarrow$ Force SHO

$$\boxed{\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = -\frac{e}{m}E(z=0,t)}$$

General solution $\chi(t) = \chi_{\text{transient}}(t) + \chi_{\text{st.}}(t)$

Steady state solution:

Charge oscillates with freq. of drive: $\chi(t) = \text{Re}(\tilde{\chi}_0 e^{-i\omega t})$

$$\Rightarrow -\omega^2 \tilde{\chi}_0 e^{-i\omega t} - i\omega\Gamma \tilde{\chi}_0 e^{-i\omega t} + \omega_0^2 \tilde{\chi}_0 e^{-i\omega t} = \frac{-e}{m} E_0 e^{-i\omega t}$$

$$\Rightarrow \tilde{\chi}_0 = \frac{-e/m\, E_0}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

Steady-state dipole moment: $p(t) = \text{Re}(\tilde{p}_0 e^{-i\omega t})$

$$\Rightarrow \tilde{p}_0 = -e\tilde{\chi}_0 = \frac{\frac{e^2}{m} E_0}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

Atomic polarizability $\tilde{p}_0 = \tilde{\alpha}(\omega) E_0$

$$\Rightarrow \boxed{\tilde{\alpha}(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma}}$$

Physically: $p(t) = \text{Re}(\tilde{\alpha}(\omega) E_0 e^{-i\omega t})$

$$\tilde{\alpha}(\omega) = \alpha'(\omega) + i\alpha''(\omega) \quad (\text{Real + imag parts})$$

$$\Rightarrow p(t) = \text{Re}(\tilde{\alpha}(\omega)) E_0 \cos\omega t + \text{Im}(\tilde{\alpha}(\omega)) E_0 \sin\omega t$$

$\Rightarrow$ $\text{Re}(\tilde{\alpha})$: Amplitude of response <u>in-phase</u> with the drive

$\text{Im}(\tilde{\alpha})$: Amplitude of response <u>in-quadrature</u> (90° out of phase) with drive

Near resonance $\omega \approx \omega_0$, Define "detuning" $\Delta = \omega - \omega_0$

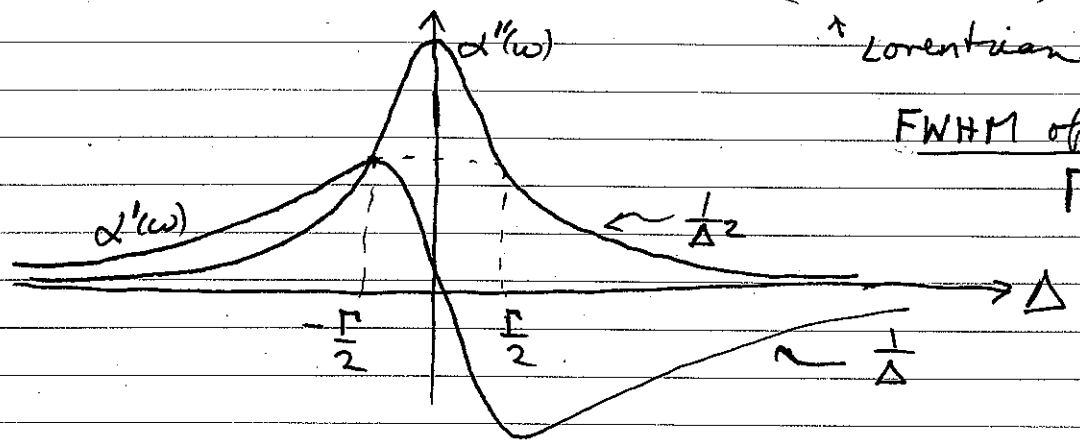
$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) = (2\omega_0 + \Delta)(-\Delta)$$

$$\approx -2\omega_0 \Delta \qquad \Delta \ll \omega_0$$

$$\Rightarrow \tilde{\alpha}(\omega) \approx \left(\frac{e^2}{2m\omega_0}\right)\left(\frac{-1}{\Delta + i\frac{\Gamma}{2}}\right) \quad \text{Complex Lorentzian}$$

$$= \frac{e^2}{2m\omega_0}\left(\frac{-\Delta}{\Delta^2 + \frac{\Gamma^2}{4}} + i\frac{\Gamma/2}{\Delta^2 + \frac{\Gamma^2}{4}}\right)$$

↑ Lorentzian



FWHM of Lorentzian Γ

Given N $\frac{\text{molecules}}{\text{volume}}$, Z electrons/molecule,

multiple resonances ω_j, Γ_j, f_j $\frac{\text{electrons}}{\text{molecule}}$ contribute

Total polarization: $\tilde{P}(\omega) = \sum_j N f_j \tilde{p}_j(\omega)$ (ignoring local-field corrections)

$$= \left[\frac{Ne^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}\right] \tilde{E}(\omega)$$

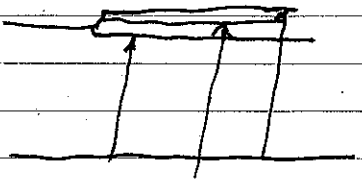
Complex susceptibility $\tilde{\chi}_e(\omega) = \sum_j N \tilde{\alpha}_j(\omega)$

"Oscillator strength f_j"

Sum rule $\sum_j f_j = Z$

Actually in classical physics f_j are a "fudge factor"

Aside



Quantum energy levels

$$f_j = \frac{|\langle \psi_{excited} | x | \psi_{ground} \rangle|^2}{2m\omega_j/\hbar}$$

$$f_j = \frac{|\langle e|x|g\rangle|^2}{\Delta x^2_{SHO}}$$

Complex dielectric constant

$$\tilde{\epsilon}(\omega) = 1 + 4\pi \tilde{\chi}_e(\omega) = 1 + 4\pi N \tilde{\alpha}(\omega)$$

$$= 1 + \frac{4\pi N e^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}$$

$\Rightarrow$ Complex index of refraction

$$\tilde{n}(\omega) = \sqrt{\tilde{\epsilon}(\omega)} = n(\omega) + i\,K(\omega) = \sqrt{1 + 4\pi \tilde{\chi}_e(\omega)}$$

$\Rightarrow$ Complex wave number $\tilde{k} = \tilde{n}(\omega)\frac{\omega}{c}$

$k_R = n_R \frac{\omega}{c}$ $\Rightarrow$ Real $(\tilde{n})$ $\Rightarrow$ Propagation

$$v_p = \frac{\omega}{k_R} = \frac{c}{n_R(\omega)}$$

$k_I = n_I \frac{\omega}{c}$ $\Rightarrow$ Im $(\tilde{n})$ $\Rightarrow$ absorption

Physically: Why $\mathrm{Im}(\tilde{\chi}) \Rightarrow$ Absorption

Steady state polarization: $\vec{P}(t) = \mathrm{Re}\left(\tilde{P}\, e^{-i\omega t}\right)$

$$E = \mathrm{Re}\left(E_0\, e^{-i\omega t}\right) = E_0 \cos\omega t \quad \Rightarrow \quad P(t) = \mathrm{Re}\left(\chi_e E_0\, e^{-i\omega t}\right)$$

(E_0: take real)

$$\Rightarrow P(t) = \mathrm{Re}(\tilde{\chi})\, E_0 \cos\omega t + \mathrm{Im}(\tilde{\chi})\, E_0 \sin\omega t$$

Polarization current: $J = \frac{\partial P}{\partial t} = \omega\left(-\mathrm{Re}(\tilde{\chi})\, E_0 \sin\omega t + \mathrm{Im}(\tilde{\chi})\, E_0 \cos\omega t\right)$

Ohmic dissipation: Rate of work done by fields on charges / volume $\vec{J}\cdot\vec{E}$

$$\Rightarrow \langle J(t)\, E(t)\rangle = \langle J(t)\, E_0 \cos\omega t\rangle = \frac{1}{2}\omega\, \mathrm{Im}(\tilde{\chi})\, E_0^2$$

$\Rightarrow$ In quadrature component of $P(t) \Rightarrow$ In phase component of J

Using complex notation:

$$\langle JE\rangle = \frac{1}{2}\mathrm{Re}\left(\tilde{J}^*\tilde{E}\right) = \frac{1}{2}\mathrm{Re}\left(-i\omega\,\tilde{P}^*\tilde{E}\right)$$

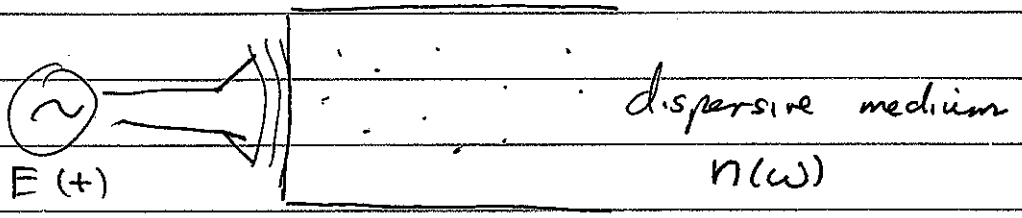
$$= \frac{\omega}{2}|E|^2\,\mathrm{Re}(-i\tilde{\chi}) = \frac{\omega}{2}\,\mathrm{Im}(\tilde{\chi})\,E_0^2$$

$\mathrm{Im}(\tilde{\chi}) > 0 \Longrightarrow$ Absorption (Passive)

$\mathrm{Im}(\tilde{\chi}) < 0 \Rightarrow$ Gain! (Laser) (Active)

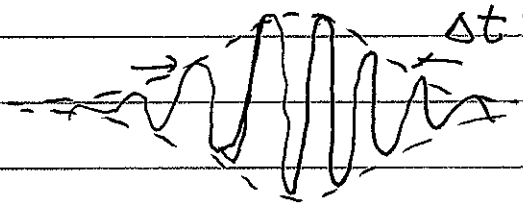
Propagation a dispersion

Suppose we inject a signal $\vec{E}(t)$ at the input face of some medium (semi-infinite)



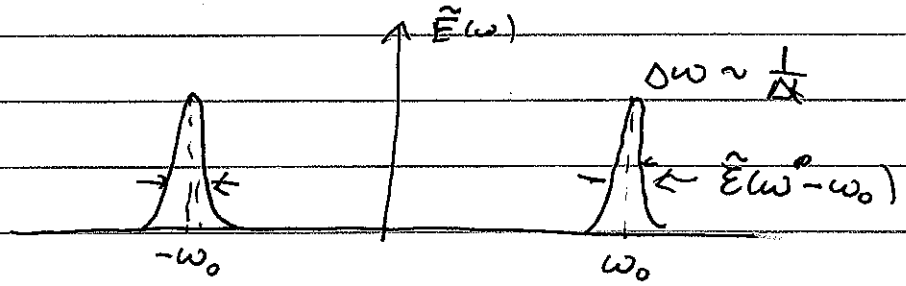
Input signal $E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{-i\omega t}\, \frac{d\omega}{2\pi}$

Quasi-monochromatic pulse $E(t) = \mathcal{E}(t)\cos\omega_0 t$



$\mathcal{E}(t)$: Slowly varying envelope

$$\tilde{E}(\omega) = \frac{1}{2}\left(\tilde{\mathcal{E}}(\omega-\omega_0) + \tilde{\mathcal{E}}(\omega+\omega_0)\right)$$



Long pulse ⇒ narrow bandwidth ⇒ $\Delta\omega \gg \omega_0$

Question: How does the signal propagate inside the medium?

Input Signal: $E(z=0,t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{i\omega t}\, \frac{d\omega}{2\pi}$

Propagating Signal: $E(z,t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{i(k(\omega)z - \omega t)}\, \frac{d\omega}{2\pi}$

$z > 0$

$k(\omega) = n(\omega)\frac{\omega}{c}$ (Assume real)

Plug in form for slowly varying pulse:

$$E(z,t) = \frac{1}{2}\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{\mathcal{E}}(\omega-\omega_0)\, e^{i(k(\omega)z-\omega t)} + \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{\mathcal{E}}^*(\omega+\omega_0)\, e^{-i(k(\omega)z-\omega t)}\right]$$

$$\cong \mathrm{Re}\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{\mathcal{E}}(\omega-\omega_0)\, e^{i(k(\omega)z-\omega t)}\right]$$

Let $\Omega = \omega - \omega_0$ (frequency difference)

$$\Rightarrow E(z,t) \cong \mathrm{Re}\left[\left(\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\,\tilde{\mathcal{E}}(\Omega)\, e^{ik(\Omega+\omega_0)z}\, e^{-i\Omega t}\right) e^{-i\omega_0 t}\right]$$

Approximation

$\tilde{\mathcal{E}}(\Omega)$ has support at $\Omega = 0$ around a small bandwidth

$\Rightarrow$ Taylor expand $k(\Omega+\omega_0)$ around $\Omega = 0$

$$k(\omega_0+\Omega) \cong k(\omega_0) + \Omega\left.\frac{dk}{d\omega}\right|_{\omega_0} + \frac{1}{2}\Omega^2\left.\frac{d^2k}{d\omega^2}\right|_0 + \cdots$$

Keeping only first two terms:

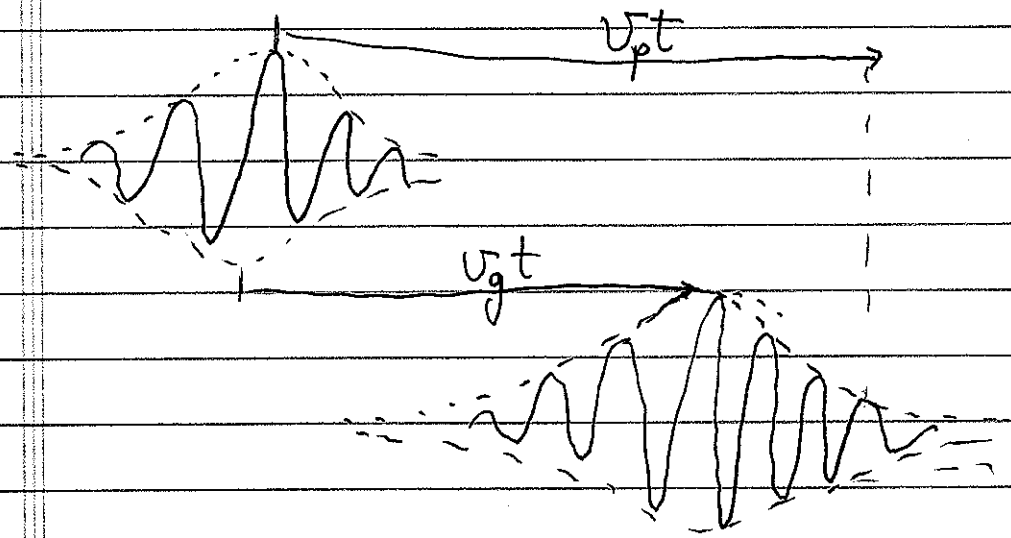
$$E(z,t) \cong \mathrm{Re}\left[\left(\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\,\tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_0} z\right)}\right) e^{i(k_0 z - \omega_0 t)}\right]$$

Define group velocity: $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$

$$\Rightarrow E(z,t) \cong \mathrm{Re}\left[\underbrace{\left(\int_{-\infty}^{\infty} \frac{d\Omega}{2\pi}\, \tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega\left(t-\frac{z}{v_g}\right)}\right)}_{= \mathcal{E}\left(t-\frac{z}{v_g}\right)}\; e^{i(k_0 z - \omega_0 t)}\right]$$

$$E(z,t) = \mathcal{E}\left(t-\frac{z}{v_g}\right) e^{i(k_0 z - \omega_0 t)}$$

$$k_0 = n(\omega_0)\frac{\omega}{c} = \frac{\omega}{v_p} \qquad v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$$



Envelope peak moves at the group velocity.

Usually $v_g < c$, but not always.

Causality? No information contained is the position of the peak for smooth wave form.

Effect of higher order terms in Taylor series expansion of $k(\omega)$: Dispersion

$$\frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\frac{1}{v_g} = \frac{dv_g}{d\omega}\frac{1}{-v_g^2}$$ G.V.D. (Group velocity dispersion)

$$E(z,t) \cong \mathrm{Re}\left[\int_{-\infty}^{\infty}\left(\frac{d\Omega}{2\pi}\,\tilde{\mathcal{E}}(\Omega)\, e^{-i\Omega\left(t-\frac{z}{v_g}\right)}\, e^{-i\frac{k''\Omega^2}{2}z}\right) e^{i(kz-\omega_0 t)}\right]$$

↑ quadratic dependence

For special case of a Gaussian input pulse output is also Gaussian with pulse duration

$$\Delta t(z) = \sqrt{\Delta t^2(0) + \left(\frac{k_0''}{\Delta t(0)}\right)^2 z^2}$$

Dispersion length: $\frac{\Delta t(0)}{k_0''}$

General argument: Due to finite bandwidth $\Delta\omega$ there will be a spread of group velocities $\Delta v_g = \frac{dv_g}{d\omega}\Delta\omega$

$\Rightarrow$ Spread of arrival times:

$$\Delta t(z) = \Delta\left(\frac{z}{v_g}\right) = z\left(\frac{-\Delta v_g}{v_g^2}\right) = -z\frac{dv_g}{d\omega}\frac{\Delta\omega}{v_g^2}$$

$$= -k''z\,\Delta\omega) \quad \text{But } \Delta\omega \sim \frac{1}{\Delta t_0}$$

Add to initial spread in square

$$\Delta t(z)^2 = \Delta t_0^2 + \frac{k''^2}{\Delta t_0^2}z^2$$ ✓