Lecture 18 Radiation by prescribed sources

Inhomogeneous wave equation (scalar field)

$$\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box} \psi(\vec{x},t) = -4\pi s(\vec{x},t)$$

$\Box$ = d'Alembertian

$s(\vec{x},t)$: Source function

General solution: $\psi(\vec{x},t) = \psi_{\text{Homogeneous}}(\vec{x},t) + \psi_{\text{Particular}}(\vec{x},t)$

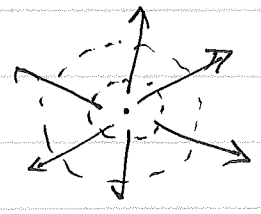
Linear response theory:

$$\psi_{\text{part}}(\vec{x},t) = \int d^3x' dt' \; G(\vec{x}-\vec{x}', t-t') \, s(\vec{x}',t')$$

$G(\vec{x},\tau)$: Greens function of wave eqn.

$\Box G = \delta^{(3)}(\vec{x})\,\delta(t)$: Wave generated by an impulse at the origin

Drop a pebble in a pond:



Spherical wave in 3D!

General source $s(\vec{x},t) = \int d^3x' dt' \; \underbrace{\delta^{(3)}(\vec{x}-\vec{x}')\,\delta(t-t')}\, s(\vec{x}',t')$

$\Downarrow$ Superposition of impulses $\Downarrow$

$\psi(\vec{x},t) = \int d^3x' dt' \; G(\vec{x}-\vec{x}', t-t')\, s(\vec{x},t')$

$\Downarrow$

Total wave

Solving for the Green's function

But taking the Fourier Transform, a differential equation becomes an algebraic equation. We can then solve for the Greens function in fourier space and inverse-Fourier to get it in space-time.

$$\tilde{G}(\vec{k},\omega) = \int d^3X\, d\tau \;\; G(\vec{X},\tau)\, e^{-i(\vec{k}\cdot\vec{X}-\omega\tau)}$$

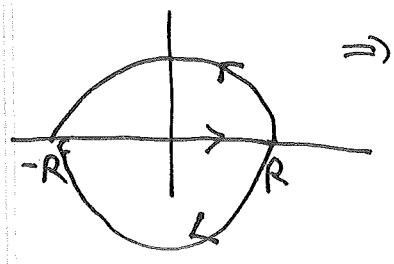
$$\Box\, G(\vec{X},\tau) = \delta^{(4)}(\vec{X},\tau) \iff \left(-k^2+\frac{\omega^2}{c^2}\right)\tilde{G}(\vec{k},\omega) = 1$$

$$\Rightarrow \quad \tilde{G}(\vec{k},\omega) = \frac{1}{\frac{\omega^2}{c^2}-k^2} = \frac{c^2}{(\omega+ck)(\omega-ck)}$$

Invert: $$G(\vec{X},\tau) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\; \tilde{G}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{X}-\omega\tau)}$$

To perform the integral we will use the tricks of contour integration on the complex plane

Aside: Let $$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, f(\omega)\, e^{-i\omega t}$$



$$\Rightarrow f(t) = \begin{cases} \lim_{R\to\infty} \oint \frac{d\tilde{\omega}}{2\pi} f(\tilde{\omega})\, e^{-i\tilde{\omega}t} & t<0 \\ \oint \frac{d\tilde{\omega}}{2\pi} f(\tilde{\omega})\, e^{-i\tilde{\omega}t} & t>0 \end{cases}$$

$$\Rightarrow f(t) = \begin{cases} 2\pi i \sum \text{Residues in upper } \tilde{\omega} \text{ plane} \\ -2\pi i \sum \text{" " lower " "} \end{cases}$$

For our problem

$$G(\vec{X},\tau) = \frac{c^2}{(2\pi)^4} \int d^3k \, e^{i\vec{k}\cdot\vec{X}} \int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\omega\tau}}{(\omega+ck)(\omega-ck)}$$

There are two poles: $\omega = \pm ck$ on <u>real axis</u>.

Thus, the causality of $G(\vec{X},\tau)$ not set but <u>imposed</u> by physical consideration.

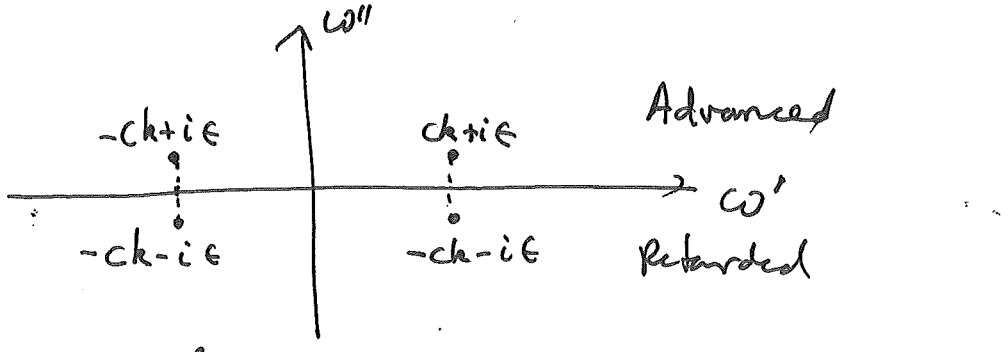
Define: <u>Retarded Green's function</u> $G^{(+)}(\vec{X},\tau) = 0 \quad \tau < 0$

<u>Advanced Greens function</u> $G^{(-)}(\vec{X},\tau) = 0 \quad \tau > 0$

For causality, Choose retarded Green's function

$\Rightarrow$ $\tilde{G}^{(+)}(\vec{k},\omega)$ must have poles only in lower $\tilde{\omega}$ plane

$\Rightarrow$ Must poles off real axis, then take limit



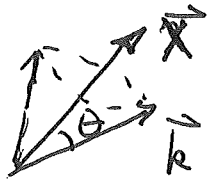
limit as $\epsilon \to 0$

$$\Rightarrow G^{(+)}(\vec{X},\tau) = \lim_{\epsilon\to 0} \frac{c^2}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{X}} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{(\omega+ck+i\epsilon)(\omega-ck+i\epsilon)}$$

$$\Rightarrow G^{(+)}(\vec{X},\tau) = \begin{cases} 0 & t<0 \\ \frac{c^2}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{X}} \left\{(-2\pi i)\left(\frac{e^{-ickt}}{2ck} + \frac{e^{ickt}}{-2ck}\right)\right\} & t>0 \end{cases}$$

$\therefore$ for $t>0$ $\quad G_R(\vec{X},\tau) = -\frac{c}{(2\pi)^3}\int \underbrace{d^3k\, \frac{\sin(ck\tau)}{k}\, e^{ik\vec{X}}}_{\equiv \mathcal{I}}$

Use Spherical Symmetry



$$d^3k = k^2\, dk\, d\Omega_k$$

$$d\Omega_k = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi \qquad \mu = \cos\theta$$

$$\Rightarrow \mathcal{I} = \int d^3k\, \frac{\sin(ck\tau)}{k}\, e^{i\vec{k}\cdot\vec{X}} = \int_0^\infty dk\, k\sin(ck\tau)\int d\Omega_k\, e^{ikr\cos\theta}$$

Aside: $\int d\Omega_k\, e^{ikr\cos\theta} = 2\pi\int_{-1}^{1} d\mu\, e^{i\mu kr} = \frac{2\pi}{ikr}\left(e^{ikr} - e^{-ikr}\right)$

$$\Rightarrow \mathcal{I} = \frac{-\pi}{r}\int_0^\infty dk \left(e^{ickt} - e^{-ickt}\right)\left(e^{ikr} - e^{-ikr}\right)$$

$$= +\frac{\pi}{r}\int_0^\infty dk \left[e^{ik(r-ct)} + e^{-ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)}\right]$$

$$\Rightarrow \mathcal{I} = \frac{\Pi}{r}\int_{-\infty}^{\infty} dk \left[e^{ik(r-c\tau)} - e^{ik(r+c\tau)}\right]$$

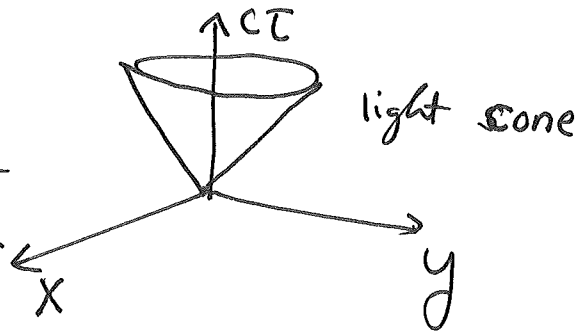
$$= \frac{\Pi}{r}\left(2\pi\delta(r-c\tau) - 2\pi\delta(r+c\tau)\right)$$

$\to 0$ since $r>0$, $\tau>0$

$$\therefore \boxed{G^{(+)}(\vec{x},\tau) = -\frac{c}{4\pi|\vec{x}|}\delta(|\vec{x}| - c\tau) = -\frac{1}{4\pi}\frac{\delta(\tau - \frac{|\vec{x}|}{c})}{|\vec{x}|}}$$

Space time diagram

Spherical wave front emiating from origin

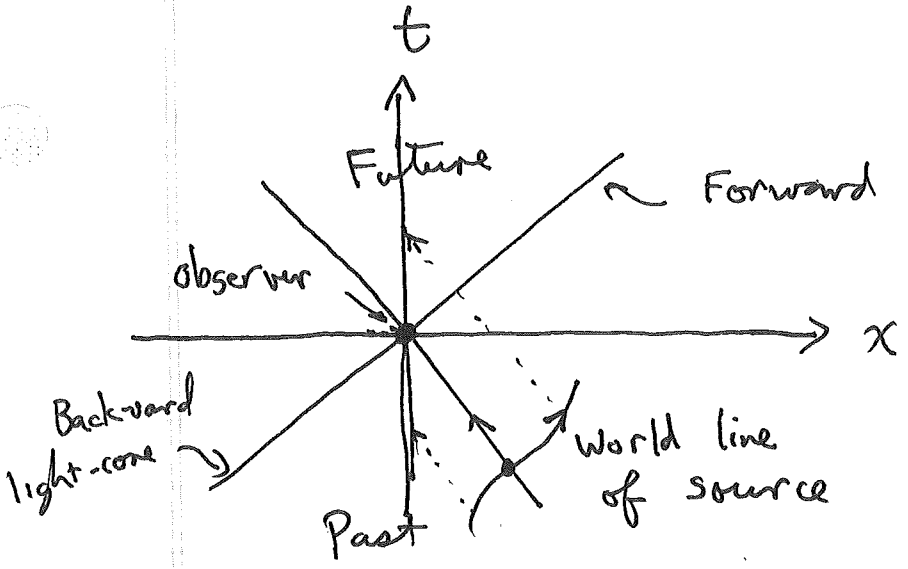


$$\Rightarrow \psi(\vec{x},t) = \int d^3x'\,dt'\; G^{(+)}(\vec{x}-\vec{x}', t-t')\,\mathcal{I}(\vec{x}',t')$$

$$= \frac{1}{4\pi}\int d^3x'\,dt'\;\delta(t-t'-\frac{|\vec{x}-\vec{x}'|}{c})\frac{\mathcal{I}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \psi(\vec{x},t) = -\frac{1}{4\pi}\int d^3x'\;\frac{\mathcal{I}(\vec{x}', t_{ret}(\vec{x}-\vec{x}';t))}{|\vec{x}-\vec{x}'|}$$

Retarded time: $t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}$ = local time − propagation time from $\vec{x} \to \vec{x}'$



Influences at any local position must have crossed my backward light cone

Example: Sinusoidally oscillating source $\rho(\vec{x},t) = \tilde{\rho}(\vec{x})\, e^{-i\omega t}$

$$\Rightarrow \psi(\vec{x},t) = -\frac{1}{4\pi}\int \rho(\vec{x}')\, \frac{e^{-i\omega\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= \underbrace{\left(-\frac{1}{4\pi}\int \rho(\vec{x}')\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\right)}_{\tilde{\psi}(\vec{x})}\, e^{-i\omega t}$$

$\tilde{\psi}(\vec{x})$ = Superposition of spherical wave emanating from $\vec{x}'$ weighted by strength of source @ $\vec{x}'$

Potential Formulation & Gauge Transformations

Maxwell's Eqns:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}}$$

Gauge invariance: $\boxed{\vec{A} \to \vec{A} + \vec{\nabla}\Lambda}$ $\quad \phi \Rightarrow \phi'$

$$\vec{E} \Rightarrow -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \cancel{\vec{\nabla}}\,\vec{\nabla}\left(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\right)$$

$$\Rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

$$\boxed{\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}}$$

Lorentz Gauge: $\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0}$

Coulomb Gauge $\boxed{\vec{\nabla}\cdot\vec{A} = 0}$

Always possible

Dynamical equations

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\frac{\partial\phi}{c\partial t}\right)$$

$$\Rightarrow \boxed{\Box\vec{A} = -\frac{4\pi}{c}\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})}$$

Lorentz Gauge

$$\boxed{\begin{aligned}\Box\vec{A} &= -\frac{4\pi}{c}\vec{J}\\ \Box\phi &= -4\pi\rho\end{aligned}} \Rightarrow \boxed{\begin{aligned}\vec{A}(\vec{x},t) &= \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\\ \phi(\vec{x},t) &= \int d^3x' \frac{\rho(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\end{aligned}}$$

Coulomb Gauge

$$\begin{aligned}\Box\vec{A} &= -\frac{4\pi}{c}\vec{J}_T\\ \nabla^2\phi &= -4\pi\rho\end{aligned} \Rightarrow \left\{\begin{aligned}\vec{A}(\vec{x},t) &= \frac{1}{c}\int d^3x' \frac{\vec{J}_T(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\\ \phi(\vec{x},t) &= \int d^3x' \frac{\rho(\vec{x}', t)}{|\vec{x}-\vec{x}'|}\end{aligned}\right\}$$

$\vec{J}_T$ is the transverse part of $\vec{J}$ (i.e. part with zero divergence)

$$\vec{J}_T = \vec{J} + \frac{1}{4\pi}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \Rightarrow \vec{\nabla}\cdot\vec{J}_T = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$