

Lecture #19a: Multipole Radiation: Electric Dipole Radiation

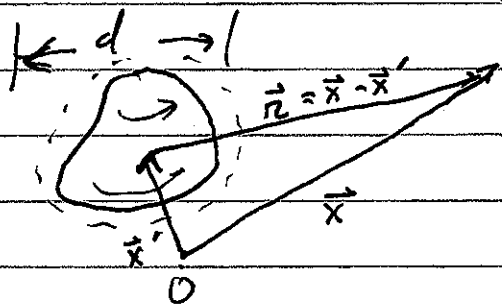
In the Lorentz Gauge: $\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$

$$\Rightarrow \square \vec{A} = -\frac{4\pi}{c} \vec{J}, \quad \square \phi = -4\pi \rho$$

$$\Rightarrow \boxed{\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3x' \frac{\vec{J}(\vec{x}', t_{\text{ret}})}{|\vec{x} - \vec{x}'|}, \quad \phi(\vec{x}, t) = \int d^3x' \frac{\rho(\vec{x}', t_{\text{ret}})}{|\vec{x} - \vec{x}'|}}$$

$$t_{\text{ret}} = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

Given localized, slowly varying source
 ↑ (we will quantify this)



Characteristic time scale for change in sources: T
 (e.g. $\omega \sim \frac{1}{T}$)

Multipole expansion as in states: $d \ll r$

$$|\vec{x} - \vec{x}'| \approx r - \hat{r} \cdot \vec{x}', \quad \frac{1}{|\vec{x} - \vec{x}'|} \approx \frac{1}{r} + \frac{\hat{r} \cdot \vec{x}'}{r^2}$$

Slowly varying source $\frac{d}{c} \ll T$ (Time to propagate across source $\ll$ time scale of change of source)

$$\text{If } T \sim \frac{1}{\omega} \Rightarrow d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi} \quad \boxed{d \ll \lambda}$$

$$\Rightarrow t_{\text{ret}} = t - \frac{|\vec{x} - \vec{x}'|}{c} = t - \underbrace{\frac{r}{c}}_{\text{Retarded time from } 0: t_0} + \underbrace{\frac{\hat{r} \cdot \vec{x}'}{c}}_{\text{Extra time of propagation}}$$

Retarded time from 0: t_0

Extra time of propagation

Given slowly varying function localized to small $\vec{x}'$

$$f(t_{\text{ret}}) = f(t_0 + \hat{r} \cdot \frac{\vec{x}'}{c} + \dots)$$

Taylor series $\approx f(t_0) + \frac{\hat{r} \cdot \vec{x}'}{c} \dot{f}(t_0) + \dots$

$$\Rightarrow \phi(\vec{x}, t) = \int d^3x' \left(\frac{1}{r} + \frac{\hat{r} \cdot \vec{x}'}{r^2} + \dots \right) \left(\rho(\vec{x}', t_0) + \frac{\hat{r} \cdot \vec{x}'}{c} \dot{\rho}(\vec{x}', t_0) + \dots \right)$$

$$\vec{A}(\vec{x}, t) = \int d^3x' \left(\frac{1}{r} + \frac{\hat{r} \cdot \vec{x}'}{r^2} + \dots \right) \left(\vec{J}(\vec{x}', t_0) + \frac{\hat{r} \cdot \vec{x}'}{c} \dot{\vec{J}}(\vec{x}', t_0) + \dots \right)$$

Dipole Radiation

Lowest
non

$$\phi = \int d^3x' \frac{\rho(\vec{x}', t_0)}{r} + \frac{\hat{r} \cdot \int \vec{x}' \rho(\vec{x}', t_0) d^3x'}{r^2} + \frac{\hat{r}}{cr} \int \vec{x}' \dot{\rho}(\vec{x}', t_0) d^3x' + \dots$$

$$\Rightarrow \phi = \underbrace{\frac{q}{r}} + \underbrace{\frac{\hat{r} \cdot \vec{p}(t_0)}{r^2} + \frac{\hat{r}}{cr} \cdot \dot{\vec{p}}(t_0)} + \dots$$

⊗ quasi-static

radiation and induction

$$q = \int d^3x' \rho(\vec{x}', t_0) = \text{monopole moment}$$

(independent of time for localized source)

$$\vec{p}(t_0) = \text{dipole moment at the time } t_0 = t - \frac{r}{c}$$

$$\vec{A} = \frac{1}{cr} \int d^3x' \vec{J}(\vec{x}', t_0) + \frac{\hat{r}}{cr^2} \cdot \int d^3x' \vec{x}' \vec{J}(\vec{x}', t_0)$$

$$+ \frac{\hat{r}}{c^2 r} \cdot \int \vec{x}' \dot{\vec{J}}(\vec{x}', t_0) d^3x' + \dots$$

Aside: $\int d^3x' \vec{J}(\vec{x}', t_0) = - \int d^3x \vec{x}' \underbrace{\vec{\nabla} \cdot \vec{J}(\vec{x}', t_0)}_{-\dot{\vec{p}}} = \dot{\vec{p}}(t_0)$

To lowest non-vanishing term:

$$\vec{A}(\vec{x}, t) = \frac{\dot{\vec{p}}(t_0)}{cr} + \dots$$

radiation and induction contribution

Now the fields:

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi = -\frac{\ddot{\vec{p}}}{c^2 r} - \vec{\nabla} \phi$$

Aside: $\vec{\nabla} \phi = \frac{q}{r^2} \hat{r} + \frac{3\hat{r}(\hat{r} \cdot \dot{\vec{p}}(t_0)) - \dot{\vec{p}}(t_0)}{r^3}$

$$- \frac{x_j}{r^3} \vec{\nabla} p_j(t_0) - \vec{\nabla} \left(\frac{\hat{r}}{cr} \cdot \dot{\vec{p}}(t_0) \right)$$

Double aside:

$$\vec{\nabla} p_j(t_0) = \dot{p}_j \vec{\nabla} t_0 = -\frac{\dot{p}_j}{c} \vec{\nabla} r = -\frac{\dot{p}_j}{c} \hat{r}$$

$$\Rightarrow \frac{x_j}{r^3} \vec{\nabla} p_j(t_0) = -(\hat{r} \cdot \dot{\vec{p}}(t_0)) \frac{\hat{r}}{cr^2}$$

Triple aside: $\vec{\nabla} \left(\frac{\hat{r}}{cr} \cdot \dot{\vec{p}}(t_0) \right) = \frac{\dot{\vec{p}}}{cr^2} - \frac{2(\hat{r} \cdot \dot{\vec{p}}) \hat{r}}{cr^2}$

$$- \frac{(\ddot{\vec{p}} \cdot \hat{r}) \hat{r}}{c^2 r}$$

$$\vec{E} = \frac{q}{r^2} \hat{r} + \frac{3\hat{r}(\hat{r} \cdot \vec{p}(t_0)) - \vec{p}(t_0)}{r^3}$$

(Quasistatic fields
Near field zone)

$$+ \frac{3\hat{r}(\hat{r} \cdot \dot{\vec{p}}(t_0)) - \dot{\vec{p}}(t_0)}{cr^2}$$

(Induction field
Intermediate zone)

$$= \frac{1}{c^2} \left(\frac{\ddot{\vec{p}}(t_0)}{r} - \hat{r}(\hat{r} \cdot \ddot{\vec{p}}(t_0)) \right)$$

(Radiation field
Far zone)

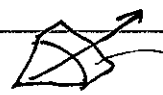
~~Radiation~~ $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\frac{\dot{\vec{p}}(t_0)}{cr} \right) = \frac{1}{cr} (\vec{\nabla} \times \dot{\vec{p}}(t_0)) - \frac{\dot{\vec{p}}(t_0)}{c} \times \vec{\nabla} \frac{1}{r}$

Aside: $(\vec{\nabla} \times \dot{\vec{p}})_i = \epsilon_{ijk} \partial_j \dot{p}_k(t_0) = \epsilon_{ijk} \dot{p}_k \partial_j(t_0)$
 $= -\epsilon_{ijk} \dot{p}_k(t_0) \frac{\hat{r}_j}{c} = \frac{\dot{\vec{p}}(t_0) \times \hat{r}}{c}$

$$\Rightarrow \vec{B}(\vec{x}, t) = \underbrace{\frac{\dot{\vec{p}}(t_0) \times \hat{r}}{cr^2}}_{\text{Induction}} + \underbrace{\frac{\ddot{\vec{p}}(t_0) \times \hat{r}}{c^2 r}}_{\text{Radiation}}$$

Radiation fields fall off like $\frac{1}{r}$

Why? Intensity $\langle \vec{S} \rangle \sim |\vec{E}|^2$

 $da = r^2 d\Omega$

Energy flow to infinity $\int \langle \vec{S} \rangle d\vec{a} \sim \int |\vec{E}|^2 r d\Omega$

Finite as $r \rightarrow \infty \Rightarrow I \sim \frac{1}{r^2} \Rightarrow E \rightarrow \frac{1}{r}$

Field Character different in different regions of space

Consider case $q_{net} = 0$

$$E \sim \frac{p}{r^3} + \frac{\dot{p}}{r^2 c} + \frac{\ddot{p}}{c^2 r} \quad \ddot{p} \sim \frac{p}{T}, \quad \ddot{p} \sim \frac{p}{T^2}$$

$$\Rightarrow E \sim \frac{p}{r^3} \left(1 + \frac{r}{cT} + \left(\frac{r}{cT} \right)^2 \right), \quad T \sim \frac{1}{\omega} \Rightarrow cT \sim \lambda$$

$$\Rightarrow E \sim \frac{p}{r^3} \left(1 + \frac{r}{\lambda} + \left(\frac{r}{\lambda} \right)^2 \right)$$

Near field $r \ll \lambda$

$$E \sim \frac{p}{r^3}, B=0$$

Quasi-static
~~Quasi~~

Intermediate $r \sim \lambda$

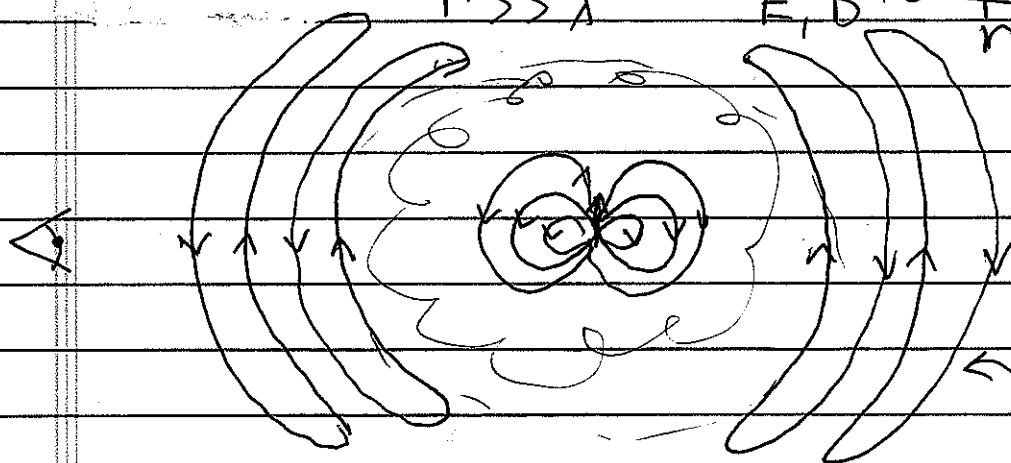
$$E, B \sim \frac{\dot{p}}{r^2}$$

Induction

$r \gg \lambda$

$$E, B \sim \frac{\ddot{p}}{r}$$

Radiation



(Electric field at some instant)

Radiation fields
"detached"
from
charge

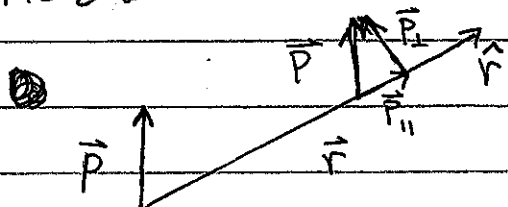
Radiation fields propagate energy to infinity. Unlike quasi-static case energy cannot be ~~recovered~~ brought back to the charge configuration

Electromagnetic dipole radiation

$$\vec{E}_{\text{rad}} = - \left(\frac{\ddot{\vec{p}}(t_0) - \hat{r}(\ddot{\vec{p}}(t_0) \cdot \hat{r})}{c^2 r} \right)$$

$$\vec{B}_{\text{rad}} = \frac{\ddot{\vec{p}}(t_0) \times \hat{r}}{c^2 r}$$

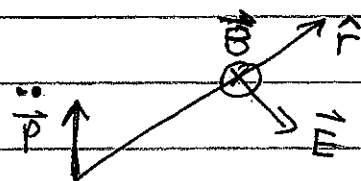
Fields are "transverse" to observation direction



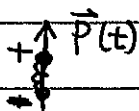
$$\vec{p}_{\parallel} = \hat{r} \cdot \vec{p}$$

$$\vec{p}_{\perp} = \vec{p} - \vec{p}_{\parallel} = \hat{r} \times (\vec{p} \times \hat{r})$$

$$\Rightarrow \begin{cases} \vec{E}_{\text{rad}} = - \frac{\ddot{\vec{p}}_{\perp}(t - \frac{r}{c})}{c^2 r} \\ \vec{B}_{\text{rad}} = \hat{r} \times \vec{E}_{\text{rad}} \end{cases}$$



Example Linearly oscillating dipole: $\vec{p} = p_0 \cos \omega t \hat{z}$



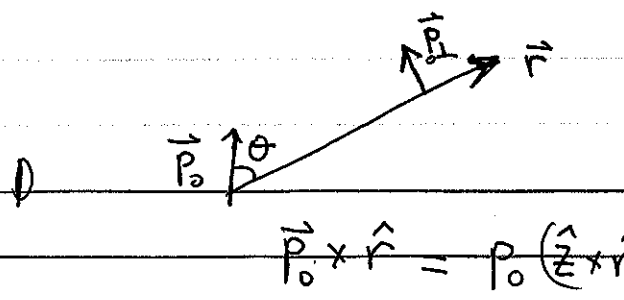
$$\vec{p}(t) = \vec{p}_0 \cos \omega t = \text{Re}(\vec{p}_0 e^{-i\omega t})$$

look at complex fields: Take real part in the end

$$\vec{E}_{\text{rad}} = - \frac{\ddot{\vec{p}}_{\perp}(t - \frac{r}{c})}{c^2 r} = \frac{\omega^2}{c^2} \frac{\vec{p}_{0\perp}}{r} e^{-i\omega(t - \frac{r}{c})}$$

$$\Rightarrow \vec{E}_{\text{rad}} = k^2 \frac{\vec{p}_{0\perp}}{r} e^{i(kr - \omega t)}$$

spherical wave
of amplitude $\vec{p}_{0\perp}$



$$\vec{p}_0 = p_0 \hat{p}$$

$$\vec{p}_{0\perp} = \hat{r} \times (\vec{p}_0 \times \hat{r})$$

$$= -p_0 \sin\theta \hat{\theta}$$

$$\vec{p}_0 \times \hat{r} = p_0 (\hat{z} \times \hat{r}) = p_0 \sin\theta \hat{\phi}$$

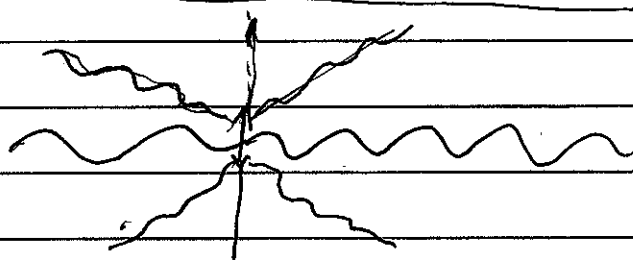
$$\Rightarrow \vec{E}_{\text{rad}}(\vec{x}, t) = -k^2 \sin\theta p_0 \frac{e^{i(kr - \omega t)}}{r} \hat{\theta}$$

$$\vec{B}_{\text{rad}}(\vec{x}, t) = \frac{\vec{p}_0 \times \hat{r}}{c^2 r} = -k^2 \sin\theta p_0 \frac{e^{i(kr - \omega t)}}{r} \hat{\phi}$$

Anisotropic transverse vector spherical wave

$$|\vec{E}| = |\vec{B}| \quad \vec{E} \perp \vec{B} \perp \vec{k} \quad \vec{B} = \hat{k} \times \vec{E}$$

$$\vec{k} = k \hat{r}$$

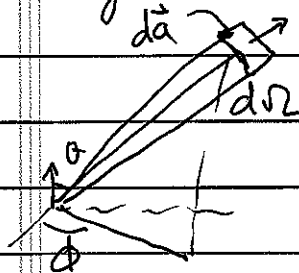


Think of anisotropy
like waves on a string

Energy flux: $\langle \vec{S} \rangle = \frac{c}{8\pi} \vec{E}_{\text{rad}}^* \times \vec{B}_{\text{rad}} = \frac{c}{8\pi} E_{\text{rad}}^2 \hat{r}$

$$\langle \vec{S} \rangle = \frac{ck^4}{8\pi} p_0^2 \frac{\sin^2\theta}{r^2} \hat{r} \propto \frac{1}{r^2}$$

Angular distribution of radiated power



$$d\vec{a} = \hat{r} r^2 d\Omega \Rightarrow dP_{\text{Power}} = \langle \vec{S} \rangle \cdot d\vec{a}$$

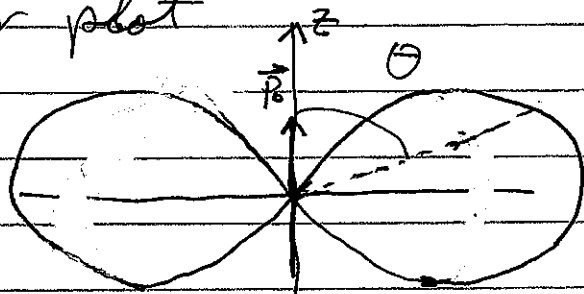
Power radiated per unit solid
angle

$$\frac{dP_{\text{Power}}}{d\Omega} = \langle \vec{S} \rangle \cdot r^2 \hat{r}$$

For linear oscillating dipole:

$$\frac{d\text{Power}}{d\Omega} = \frac{ck^4}{8\pi} p_0^2 \sin^2 \theta$$

Polar plot



Dipole radiation pattern
(independent of ϕ)

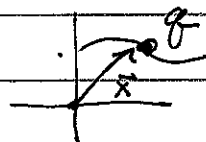
Total power radiated into all directions

$$P_{\text{Power}} = \int \frac{dP}{d\Omega} d\Omega = \frac{ck^4}{8\pi} p_0^2 \int \sin^2 \theta d\Omega \quad \xrightarrow{\frac{8\pi}{3}}$$

$$\Rightarrow \boxed{P_{\text{Power}} = \frac{ck^4}{3} p_0^2}$$

Larmor's formula

For a single moving charge



$$\vec{p}(t) = q \vec{x}(t) \Rightarrow \vec{E}_{\text{rad}} = -\frac{q \ddot{\vec{x}}(t - \frac{r}{c})}{c^2 r}$$

$$\Rightarrow \boxed{\vec{E}_{\text{rad}} = -\frac{q}{c^2 r} \vec{a}(t - \frac{r}{c})}$$

Radiated field proportional
to acceleration of charge

Instantaneous total power radiated

$$\boxed{P_{\text{Power}}(t) = \frac{2}{3} \frac{q^2}{c^3} |\vec{a}(t - \frac{r}{c})|^2}$$