Physics 511

Problem Set #1 Solutions

(1) Vector identities

(a) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) \equiv \vec{V}\cdot\vec{W} = V_i W_i$

where $V_i = (\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k$, $W_i = (\vec{C}\times\vec{D})_i = \epsilon_{ilm} C_l D_m$

(Remember, repeated indicies are summed over. Never use the same dummy index for two different sums)

$$\Rightarrow (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \epsilon_{ijk}\epsilon_{ilm} A_j B_k C_l D_m = \epsilon_{jki}\epsilon_{ilm} A_j B_k C_l D_m$$

cyclic permutation

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) A_j B_k C_l D_m$$

$$= A_j B_k C_j D_k - A_j B_k C_k D_j$$

$$\boxed{(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})}$$

(b) $\vec{\nabla}\times(\Psi\vec{A}) \equiv \vec{V}$

(product rule of derivatives)

$$\Rightarrow V_i = \epsilon_{ijk}\partial_j(\Psi A_k) = \epsilon_{ijk}\left(A_k(\partial_j\Psi) + \Psi(\partial_j A_k)\right)$$

$$= -\epsilon_{ikj} A_k \partial_j \Psi + \Psi\,\epsilon_{ijk}\partial_j A_k$$

non-cyclic permutation of ijk

$$\Rightarrow \boxed{\vec{V} \equiv \vec{\nabla}\times(\Psi\vec{A}) = -(\vec{A}\times\vec{\nabla})\Psi + \Psi(\vec{\nabla}\times\vec{A})}$$

(1) continued

(c) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \equiv \vec{V}$

$$\Rightarrow V_i = \underset{\underset{\nabla_j}{\Uparrow}}{\epsilon_{ijk}\,\partial_j}\Big(\underbrace{\epsilon_{k\ell m}\,\partial_\ell A_m}_{\underset{(\vec{\nabla}\times\vec{A})_k}{\Uparrow}}\Big) = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\,\partial_j\partial_\ell A_m$$

$$= \partial_j\partial_i A_j - \partial_j\partial_j A_i = \partial_i(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A_i$$

$$\therefore \boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}}$$

What about the "old fashion" way, proving by explicitly evaluation of curl, div, grad in Cartesian coordinates

Let $\vec{F} = \vec{\nabla}\times\vec{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x)$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \hat{x}(\partial_y F_z - \partial_z F_y) + \hat{y}(\partial_z F_x - \partial_x F_z) + \hat{z}(\partial_x F_y - \partial_y F_x)$$

$$= \hat{x}(\partial^2_{xy}A_y - \partial_y^2 A_x - \partial_z^2 A_x - \partial^2_{xz}A_z)$$
$$+ \hat{y}(\partial^2_{yz}A_z - \partial_z^2 A_y - \partial_x^2 A_y + \partial^2_{xy}A_x)$$
$$+ \hat{z}(\partial^2_{zx}A_x - \partial_x^2 A_z - \partial_y^2 A_z + \partial^2_{zy}A_y)$$

$$= \hat{x}\,\partial_x(\partial_y A_y + \partial_z A_z) + \hat{y}\,\partial_y(\partial_x A_x + \partial_z A_z) + \hat{z}\,\partial_z(\partial_x A_x + \partial_y A_y)$$
$$- (\partial_y^2 + \partial_z^2)A_x\,\hat{x} \quad (\partial_x^2 + \partial_z^2)A_y\,\hat{y} \quad (\partial_x^2 + \partial_y^2)A_z\,\hat{z}$$

Now:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)(\partial_x A_x + \partial_y A_y + \partial_z A_z)$$
$$- (\partial_x^2 + \partial_y^2 + \partial_z^2)(A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$

The terms $\partial_x^2 A_x$, $\partial_y^2 A_y$, $\partial_z^2 A_z$ cancel, reproducing the the result above. OK, which method is easier?

(2) Irreducible tensors

(a) Consider the second rank tensor $U_{ij} = A_i B_j$ (outer product)

This can be decomposed into a sum of "irreducible tensors"

$$U_{ij} = U_{ij}^{(0)} + U_{ij}^{(1)} + U_{ij}^{(2)}$$

$$U_{ij}^{(0)} \equiv \frac{1}{3} \mathrm{Tr}(U)\, \delta_{ij} = \frac{1}{3}\left(\sum_i U_{ii}\right) \delta_{ij} = \frac{1}{3}(A_i B_i)\, \delta_{ij} = \boxed{\frac{1}{3}(\vec{A}\cdot\vec{B})\, \delta_{ij}}$$

$$U_{ij}^{(1)} \equiv \frac{1}{2}(U_{ij} - U_{ji}) = \frac{1}{2}(A_i B_j - A_j B_i) = \frac{1}{2}(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) A_\ell B_m$$

$$= \frac{1}{2}\epsilon_{ijk}\epsilon_{k\ell m} A_\ell B_m = \boxed{\frac{1}{2}\epsilon_{ijk}(\vec{A}\times\vec{B})_k}$$

$$U_{ij}^{(2)} \equiv \frac{1}{2}(U_{ij} + U_{ji}) - \frac{1}{3}\mathrm{Tr}(U)\,\delta_{ij} =$$

(b) Consider the contraction of two Cartesian tensors

$$U_{ij} W_{ij} = (U_{ij}^{(0)} + U_{ij}^{(1)} + U_{ij}^{(2)})(W_{ij}^{(0)} + W_{ij}^{(1)} + W_{ij}^{(2)})$$

We must show that $U_{ij}^{(\ell)} W_{ij}^{(\ell')} = 0$ if $\ell \neq \ell'$

- $\ell = 0, \ell' = 1$ $\quad U_{ij}^{(0)} W_{ij}^{(1)} = \frac{\mathrm{Tr}(U)}{3}\delta_{ij} W_{ij}^{(1)} = \frac{\mathrm{Tr}(U)}{3} W_{ii}^{(1)}$

  $= 0$ since $W_{ii}^{(1)} = 0$ (traceless)

- $\ell = 0, \ell' = 2$ $\quad U_{ij}^{(0)} W_{ij}^{(2)} = \frac{\mathrm{Tr}(U)}{3} W_{ii}^{(2)}$

  $= 0$ since $W_{ii}^{(2)} = 0$ (Traceless by construction)

- $\ell = 1, \ell' = 2$ $\quad U_{ij}^{(1)} W_{ij}^{(2)} = \left[\frac{1}{2}(U_{ij} - U_{ji})\right]\left[\frac{1}{2}(W_{ij} + W_{ji})\right]$

  $= \frac{1}{4}\left[U_{ij} W_{ij} - U_{ji} W_{ij} + U_{ij} W_{ji} - U_{ji} W_{ji}\right]$

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Problem (2) continued

- $l' = 1, \; l' = 2$ $\quad U_{ij}^{(1)} W_{ij}^{(2)} = \frac{1}{4}\left[U_{ij}W_{ij} - U_{ji}W_{ij} + U_{ij}W_{ji} - U_{ji}W_{ji}\right]$

$$= \frac{1}{4}\left[\mathrm{Tr}(U^T W) - \mathrm{Tr}(UW) + \mathrm{Tr}(UW) - \mathrm{Tr}(U^T W)\right]$$

$$\Rightarrow U_{ij}^{(1)} W_{ij}^{(2)} = 0$$

(3) Helmholtz's Theorem

Consider $\vec{\nabla} \times \left(\vec{\nabla} \times \int_{\text{Volume}} \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \right) \equiv I$

$$I = \vec{\nabla}\left(\vec{\nabla} \cdot \int \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right) - \nabla^2 \int \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

Note that $\vec{\nabla}$ here acts on the <u>nonprimed</u> variable

let $\frac{1}{r} \equiv \frac{1}{|\vec{x}-\vec{x}'|}$: Bring differential operators inside the integral

$$\Rightarrow I = \vec{\nabla} \int \vec{V}(\vec{x}') \cdot \left(\vec{\nabla}\frac{1}{r}\right) d^3x' - \int \vec{V}(\vec{x}')\left(\nabla^2 \frac{1}{r}\right) d^3x'$$

Aside: $\nabla^2 \frac{1}{r} = \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$ (from class)

"Trick" (used often) $\vec{\nabla}\frac{1}{r} = -\frac{\vec{r}}{r^3} = -\frac{\vec{x}-\vec{x}'}{r^3} = -\vec{\nabla}'\frac{1}{r}$ ← gradient w.r.t. primed variables

$$\Rightarrow I = -\vec{\nabla} \int_{\text{Volume}} \left(\vec{V}(\vec{x}') \cdot \vec{\nabla}'\frac{1}{r}\right) d^3x' + 4\pi\,\vec{V}(\vec{x})$$

Finally we want to do an integration by parts to move $\vec{\nabla}'$ onto $\vec{V}(\vec{x}')$

Use $\vec{\nabla}' \cdot \left(f(\vec{x}')\,\vec{A}(\vec{x}')\right) = \vec{A} \cdot \vec{\nabla}' f + f\,\vec{\nabla}' \cdot \vec{A}$

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Thus: $\vec{V}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{r} = \vec{\nabla}'\cdot\left(\frac{\vec{V}(\vec{x}')}{r}\right) - \frac{\vec{\nabla}'\cdot\vec{V}(\vec{x}')}{r}$

$$\Rightarrow \int_{\text{Volume}} \vec{V}(\vec{x}')\cdot\left(\vec{\nabla}'\frac{1}{r}\right) d^3x' = \int_{\text{Volume}} -\frac{\vec{\nabla}'\cdot\vec{V}(\vec{x}')}{r}\, d^3x' + \underbrace{\oint_{S} \frac{\vec{V}(\vec{x}')}{r}\, d^3x'}_{\text{"Surface term"}}$$

where I have used the divergence theorem on the last integral

Now assume $V(\vec{x}) \to 0$ faster than $\frac{1}{r}$
Then as $S \to \infty$ the (surface integral) will go to 0

Putting it all together

$$\vec{V}(\vec{x}) = \frac{1}{4\pi}\vec{\nabla}\int_{\text{allspace}} \vec{V}(\vec{x}')\cdot\vec{\nabla}\frac{1}{r}\, d^3x' + \frac{1}{4\pi}\, I$$

$$\Rightarrow \vec{V}(\vec{x}) = \underbrace{-\vec{\nabla}\left[\frac{1}{4\pi}\int_{\substack{\text{all}\\ \text{space}}} \frac{\vec{\nabla}'\cdot\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right]}_{\vec{V}_L(\vec{x})} + \underbrace{\vec{\nabla}\times\left[\frac{1}{4\pi}\int_{\substack{\text{all}\\ \text{space}}} \frac{\vec{\nabla}'\times\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right]}_{\vec{V}_T(\vec{x})}$$

("Longitudinal", "Irrotational") part

("Transverse", "Solenoidal") part