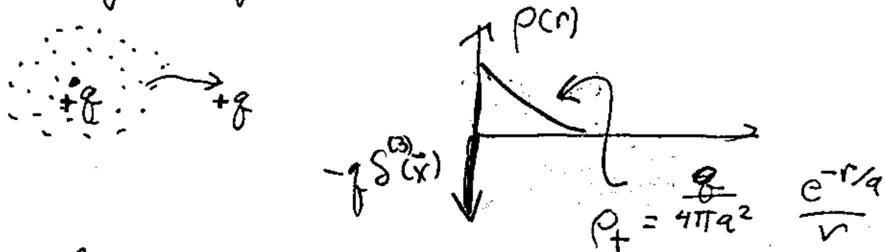
Physics 511 [redacted]

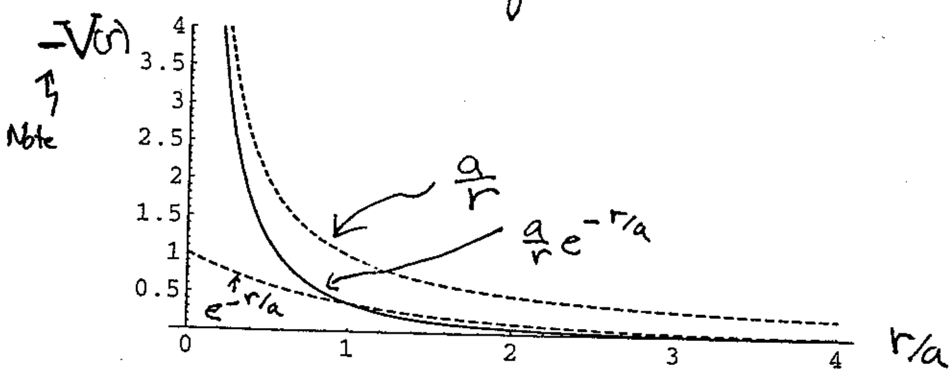
P.S. #2 Solutions

(1) The Yukawa Potential

Consider a charge distribution consisting of a point charge at the origin $-q$, and a "smeared out" charge $+q$ which "screens" the long range force of the point



(a) We want to show that this charge distribution yields the Yukawa potential $V(r) = G\,\frac{e^{-r/a}}{r/a}$



For short distances (i.e. $r \ll a$) $e^{-r/a} \approx 1$

$\Rightarrow V(r) \propto \frac{1}{r}$

For long distance (i.e. $r \gg a$) the exponential dominates

Thus, the Yukawa potential looks like a Coulomb potential for $r \ll a$, but falls off very rapidly to zero for $r \gg a$.

(b) The total charge is neutral: Proof

$$Q_{total} = -q + \int d^3r\, \rho(\vec{x}) \quad , \quad \rho(\vec{x}) = -\frac{q}{4\pi a^2}\frac{e^{-r/a}}{r}$$

(1b) continued

- $Q_{total} = -q\left[1 - \int_0^\infty \frac{e^{-r/a}}{4\pi a^2 r} 4\pi r^2 dr\right]$

Here I used the fact that ρ is spherically symmetric, so the volume element is just a spherical shell $d^3r = 4\pi r^2 dr$

Now change variables: let $u = r/a$

$$\Rightarrow Q_{total} = -q\left[1 - \int_0^\infty u e^{-u}\, du\right]$$

Using integration by parts: $\frac{d}{du}(ue^{-u}) = e^{-u} - ue^{-u}$

$$\Rightarrow \int ue^{-u}\, du = \int \left(-\frac{d}{du}(ue^{-u}) + e^{-u}\right) du$$

$$= -e^{-u} - ue^{-u} = -e^{-u}(1+u)$$

$$\therefore Q_{total} = -q\left[1 + e^{-u}(1+u)\Big|_0^\infty\right]$$

$$= -q\left[1 + 0 - 1\right] = 0$$

This makes sense physically since $V(r)$ falls off at far distances much faster than $1/r$ so there is no monopole moment. In fact this

charge distribution creates a potential which goes to zero at infinity faster than any multipole moment for a confined set of charges.

(1c) Since this charge distribution is spherically symmetric, Gauss' Law calls out to us.

First, from pure symmetry arguments, we know

$$\vec{E}(\vec{x}) = E(r)\,\hat{r}$$ (i.e. the magnitude of $\vec{E}$ depends only on r, not θ + ϕ and points in radial direction)

$\Rightarrow$ If we consider a sphere centered at the origin

$$\oint_S \vec{E}\cdot\hat{n}\,dA = 4\pi r^2 E(r)$$ where r is the radius

Gauss' Law: $= 4\pi Q_{enclosed} \Rightarrow E(r) = \dfrac{Q_{enclosed}}{r^2}$

All we need now is the charge enclosed in the sphere

$$Q_{enclosed} = \int_V \rho(\vec{r})\,d^3r = \int_0^r \rho(r')\,4\pi r'^2\,dr' \quad (r' = \text{dummy})$$

$$= -q\left(1 - \int_0^r \frac{e^{-r'/a}}{4\pi a^2 r'}\,4\pi r'^2\,dr'\right)$$

$$= -q\left(1 - \int_0^{r/a} u\,e^{-u}\,du\right) \quad (\text{same change of variables})$$

$$= -q\left(1 + e^{-u}(1+u)\Big|_0^{r/a}\right)$$

$$= -q\left(1 + e^{-r/a}\left(1+\frac{r}{a}\right) - 1\right)$$

$$\Rightarrow Q_{enc} = -q\,e^{-r/a}\left(1+\frac{r}{a}\right)$$

$$\therefore \vec{E}(r) = \frac{-q}{r^2}\left(e^{-r/a}\left(1+\frac{r}{a}\right)\hat{r}\right)$$

(Next Page)

(Ic) Continued

Now, to find the potential, use $\vec{E} = -\vec{\nabla}\Phi$

$$\Rightarrow \phi(\vec{r}) = -\int_{\vec{r}_{ground}}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{\ell}$$

where $\vec{r}_{ground}$ is the point where $= 0$ and the path of integration is arbitrary since $\vec{\nabla} \times \vec{E} = 0$

For a confined charge distribution, choose $\vec{r}_{ground} = \infty$, and since $\vec{E}$ is along the radial direction, choose $d\vec{\ell} = \hat{r}\, dr$

$$\Rightarrow \phi(\vec{r}) = -\int_{\infty}^{r} E(r')\, dr' = \int_{r}^{\infty} \frac{q}{r^2}\, e^{-r'/a}\left(1 + \frac{r'}{a}\right) dr$$

$$= \frac{-q}{a} \int_{r/a}^{\infty} du\, e^{-u}\left(\frac{1}{u^2} + \frac{1}{u}\right) \qquad \left(\text{where } u = r/a\right)$$

The integrand is a perfect derivative:

$$\frac{d}{du}\left(\frac{e^{-u}}{u}\right) = -\frac{e^{-u}}{u} - \frac{e^{-u}}{u^2} = -e^{-u}\left(\frac{1}{u^2} + \frac{1}{u}\right)$$

$$\therefore \phi(\vec{r}) = \frac{-q}{a}\left[-\frac{e^{-u}}{u}\right]_{r/a}^{\infty}$$

$$\boxed{\phi(\vec{r}) = -q\, \frac{e^{-r/a}}{r}}$$

Yukawa form with $G = -\frac{q}{a}$

1(d) The potential energy stored in the charge distribution $\rho_-(\vec{r})$

In class we found the potential energy can be expressed as

$$U = \frac{1}{8\pi}\int_{\text{all space}} |\vec{E}|^2 \, d^3\vec{x}$$

From part (c), $|\vec{E}|^2 = \frac{q^2}{r^4} e^{-2r/a} \left(1+\frac{r}{a}\right)^2$

Again, radial symmetry; choose $d^3r = 4\pi r^2 dr$

$$\Rightarrow U = \frac{q^2}{2}\int_0^\infty e^{-2r/a}\left(1+\frac{r}{a}\right)^2 \frac{dr}{r^2}$$

$$u = r/a$$

$$\Rightarrow U = \frac{q^2}{2a}\int_0^\infty e^{-2u}\left(\frac{1+u}{u}\right)^2 du$$

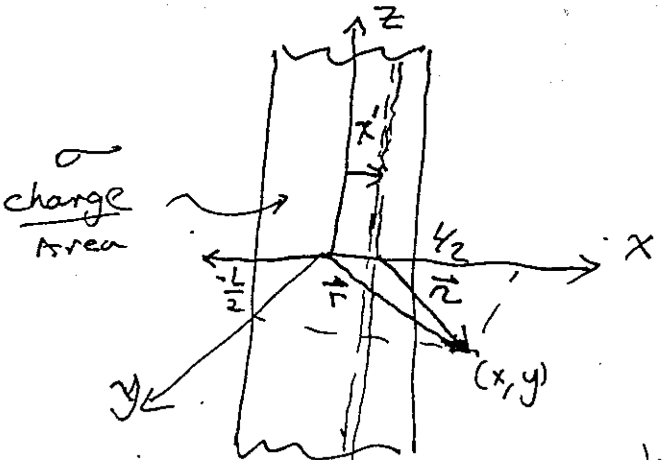
$$\Rightarrow U = \frac{q^2}{2a}\left[-\left(\frac{1}{2}+\frac{1}{u}\right)e^{-2u}\right]_0^\infty$$

$$= \frac{q^2}{2a}\left[\frac{1}{2}+\infty\right]!$$

Diverges!

Why? This is because of the point charge. The potential energy stored in a point singularity is infinite. It makes no sense to ask how much work it take to push charge q into a point

(2) Potential of a strip of charge



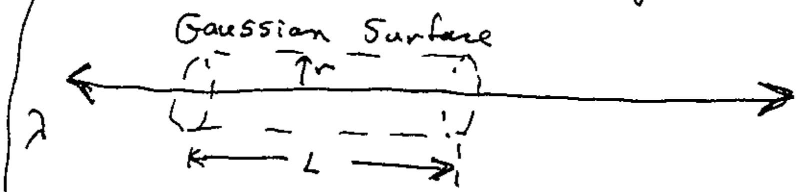
Break up the strip into substrip of thickness dx': These are infinite line charges, with charge per unit length

$$\lambda = \sigma dx'$$

We then integrate the contribution of each line charge as x' goes from $-\frac{L}{2} \to L/2$

Aside

What is the potential of an infinite line charge? This is most easily obtained from Gauss' Law:



$$\oint \vec{E} \cdot d\vec{a} = E(r) 2\pi r L = 4\pi\lambda L$$

$$\Rightarrow E(r) = \frac{2\lambda}{r}$$

Now this is tricky. The potential is defined by

$$\Phi(\vec{x}) = -\int_{\vec{r}_0}^{\vec{x}} \vec{E} \cdot d\vec{l}$$, where $\vec{r}_0$ is ground

For an infinite line charge, we cannot take $\vec{r}_0$ at ∞. However, the choice of $\vec{r}_0$ is arbitrary. Choose a radial path $dl = \hat{r}\, dr$

$$\Phi(r) = -\int_{r_0}^{r} E(r) dr = \left[-\ln(r) + \ln(r_0)\right] 2\lambda$$

For convenience, choose $r_0 = 1$ (arbitrary) $\quad \Phi(r) = -2\lambda \ln(r)$

OK, so now we have the potential of a line charge

$$\Phi(r) = -2\lambda \ln(r)$$ where r is the radial distance from the line to the point of observation.

Back to the problem at hand. The line charge at $\vec{r}'=x'\hat{x}$ contribute a potential dV at the observation point ~~to~~ $\vec{r} = x\hat{x} + y\hat{y}$ of

$$d\Phi(x,y;\, x') = -2\lambda \ln(\mathscr{r}) \quad , \quad \lambda = \sigma\, dx'$$

where $\mathscr{r} = |\vec{\mathscr{r}}| = |\vec{r} - \vec{r}'| = |(x-x')\hat{x} + y\hat{y}|$

$$= \sqrt{(x-x')^2 + y^2}$$

$$\Rightarrow d\Phi(x,y;\, x') = -2\sigma\, dx' \ln\left[\sqrt{(x-x')^2 + y^2}\right]$$

Now integrate:

$$\Phi(x,y) = \int_{x'=-L/2}^{x'=L/2} d(x,y;\, x') = -2\sigma \int_{-L/2}^{L/2} dx' \ln\left[\sqrt{(x-x')^2+y^2}\right]$$

let $u = x' - x$ $\qquad du = dx'$

$x' = u + x$

$$\Rightarrow \Phi(x,y) = -2\sigma \int_{x-L/2}^{x+L/2} du \ln\left(\left[u^2+y^2\right]^{1/2}\right) = -\sigma \int_{x-L/2}^{x+L/2} du \ln(u^2+y^2)$$

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From the integral

$$\int du \, \ln[u^2+y^2] = -2u + 2y \tan^{-1}\left(\frac{u}{y}\right) + u \ln[u^2+y^2]$$

$$\Rightarrow \Phi(x,y) = -\sigma \left[-2u + 2y\tan^{-1}\left(\frac{u}{y}\right) + u\ln[u^2+y^2]\right]_{x-L/2}^{x+L/2}$$

$$\boxed{\begin{aligned}\Phi(x,y) = -\sigma \Big[&-2L + 2y\left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right] \\ &+ (x+L/2)\ln[(x+L/2)^2+y^2] \\ &- (x-L/2)\ln[(x-L/2)^2+y^2] \Big]\end{aligned}}$$

(b) The electric field

$$\vec{E} = -\vec{\nabla}\Phi = -\hat{x}\frac{\partial}{\partial x} - \hat{y}\frac{\partial}{\partial y} = E_x\hat{x} + E_y\hat{y}$$

$$-\frac{\partial\Phi}{\partial x} = \sigma \Big\{ 2y\left[\frac{1}{y}\left(\frac{1}{1+\left(\frac{x+L/2}{y}\right)^2}\right) - \frac{1}{y}\left(\frac{1}{1+\left(\frac{x-L/2}{y}\right)^2}\right)\right]$$

$$+ \frac{2(x+L/2)^2}{y^2+(x+L/2)^2} - \frac{2(x-L/2)^2}{y^2+(x-L/2)^2}$$

$$+ \ln[(x+L/2)^2+y^2] - \ln[(x-L/2)^2+y^2]\Big\}$$

$$= \sigma\Big\{\frac{2y^2+2(x+L/2)^2}{y^2+(x+L/2)^2} - \frac{2y^2+2(x-L/2)^2}{y^2+(x-L/2)^2}$$

$$+ \ln[(x+L/2)^2+y^2] - \ln[(x-L/2)^2+y^2]\Big\}$$

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$$\therefore \quad E_x = -\frac{\partial \Phi}{\partial x} = \sigma \left\{ \ln\left[(x+L/2)^2+y^2\right] - \ln\left[(x-L/2)^2+y^2\right] \right\}$$

$$\boxed{E_x = 2\sigma \left\{ \frac{1}{2} \ln\left[\frac{(x+L/2)^2+y^2}{(x-L/2)^2+y^2} \right] \right\}}$$

$$E_y = -\frac{\partial \Phi}{\partial y} = \sigma \left\{ 2\left(\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right) \right.$$

$$+ 2y\left(-\frac{(x+L/2)}{y^2} \frac{1}{1+\left(\frac{x+L/2}{y}\right)^2} + \frac{(x-L/2)}{y^2} \frac{1}{1+\left(\frac{x-L/2}{y}\right)^2} \right)$$

$$\left. + (x+L/2)\frac{2y}{y^2+(x+L/2)^2} - (x-L/2)\frac{2y}{y^2+(x-L/2)^2} \right\}$$

$$\Rightarrow E_y = 2\sigma \left\{ \tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right) \right.$$

$$- \frac{y(x+L/2)}{y^2+(x+L/2)^2} + \frac{y(x+L/2)}{y^2+(x+L/2)^2}$$

$$\left. + \frac{y(x-L/2)}{y^2+(x-L/2)^2} - \frac{y(x-L/2)}{y^2+(x-L/2)^2} \right\}$$

$$\Rightarrow \boxed{E_y = 2\sigma \left\{ \tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right) \right\}}$$

(Next Page)

(G) limit $L \to \infty$ (i.e $x \ll L$, $y \ll L$)

$$\lim_{L\to\infty} \left[\underbrace{\tan^{-1}\left(\frac{x+L/2}{y}\right)}_{\Downarrow \; \frac{\pi}{2}} - \underbrace{\tan^{-1}\left(\frac{x-L/2}{y}\right)}_{\Downarrow \; -\pi/2} \right] = \pi$$

Now things get tricky again

$$\lim_{L\to\infty} \left\{ (x+L/2) \ln\left[(x+L/2)^2+y^2\right] - (x-L/2) \ln\left[(x-L/2)^2+y^2\right] \right)$$

$$= \lim_{L\to\infty} \left(L \ln(L/2) + L \ln(L/2) \right)$$

$$= \lim_{L\to\infty} \left[2L \ln(L/2) \right]$$

So $\displaystyle \lim_{L\to\infty} \Phi(x,y) = -2\pi\sigma y + \underbrace{\lim_{L\to\infty}\left(-2L + 2L \ln(L/2)\right)}_{?}$

The term in brackets blows up, but thats really an artifact of the problem of where to put ground. In general we can add or subtract any constant from V without effecting the physics

So $\boxed{\displaystyle \lim_{L\to\infty} \Phi(x,y) = -2\pi\sigma\, y + \text{constant}}$

Good: This is the potential of an infinite plane

$$\vec{E} = -\vec{\nabla}\Phi = 2\pi\sigma\,\hat{y} \quad (y>0)$$

(C) continued

Now take the limit as $x, y \to \infty$

- $$\lim_{x,y\to\infty} \left[\tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right)\right]$$

$$= \lim_{x,y\to\infty} \left[\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x}{y}\right)\right] = 0$$

- $$\lim_{x,y\to\infty} \left\{(x+L/2)\ln\left[(x+\tfrac{L}{2})^2+y^2\right] - (x-L/2)\ln\left[(x-L/2)^2+y^2\right]\right\}$$

$$= L\ln(x^2+y^2) = L\ln(r^2) = 2L\ln(r)$$

$$\therefore \lim_{x,y\to\infty} V(x,y) = \sigma\left(-2L + 2L\ln(r)\right)$$

$$\boxed{\lim_{x,y\to\infty} V(x,y) = -2\sigma L\ln(r) + \text{Const}}$$

Good, this is the potential of an infinite line charge with $\lambda = \sigma L$ the $\frac{\text{charge}}{\text{length}}$

```
Needs["Graphics`PlotField`"]
```

In[21]:=

```
V[x_,y_] := 2y (ArcTan[(L/2-x)/y] + ArcTan[(L/2+x)/y]) -
            (x-L/2) Log[(x-L/2)^2+y^2] + (x+L/2) Log[(x+L/2)^2+y^2]
```

In[22]:=

```
L=1  (*  Choosing L=1 to set the scale  *)
```

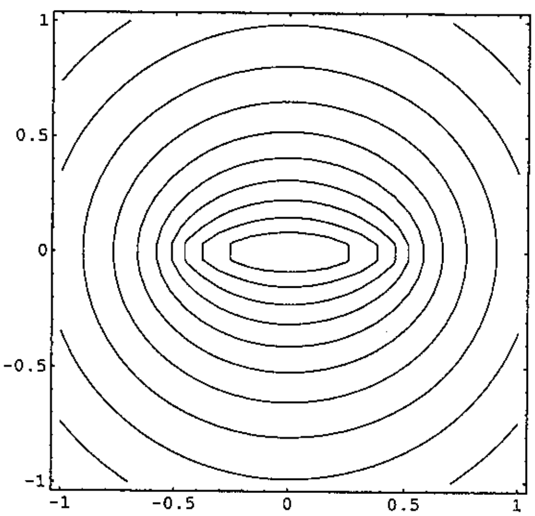
Out[22]=

```
1
```

Single Positively charged strip

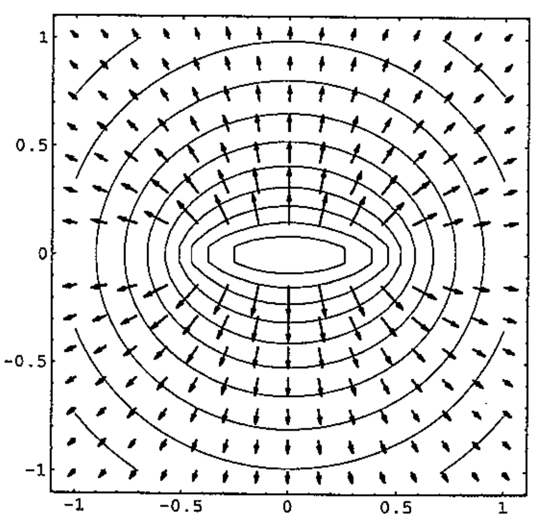
x and y on the order of L

```
ContourPlot[V[x,y], {x,-1,1}, {y,-1,1}, ContourShading->False,
            PlotPoints->30]
```



Mathematica Code

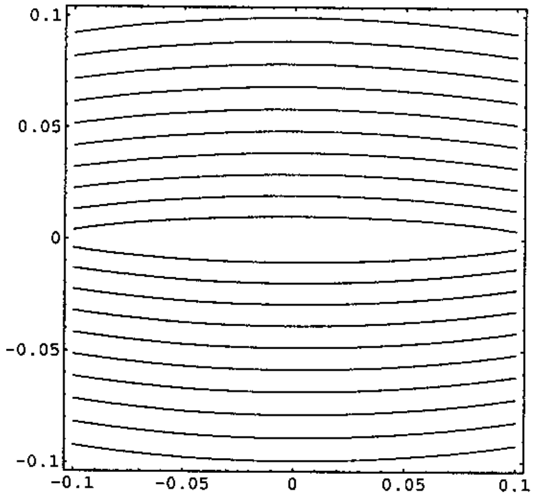
```
-ContourGraphics-
PlotGradientField[V[x,y], {x,-1,1}, {y,-1,1}]

Show[%24,%25]
```



```
-Graphics-
```

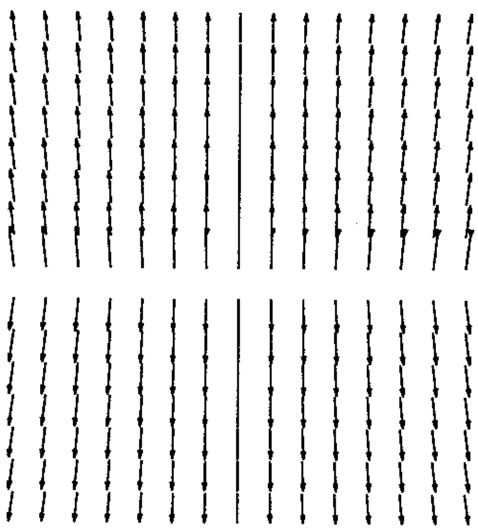
■ ~~x and y on the small compared to L (looks more like a plane)~~

```
ContourPlot[V[x,y], {x,-.1,.1}, {y,-.1,.1}, ContourShading->False,
             PlotPoints->30]
```



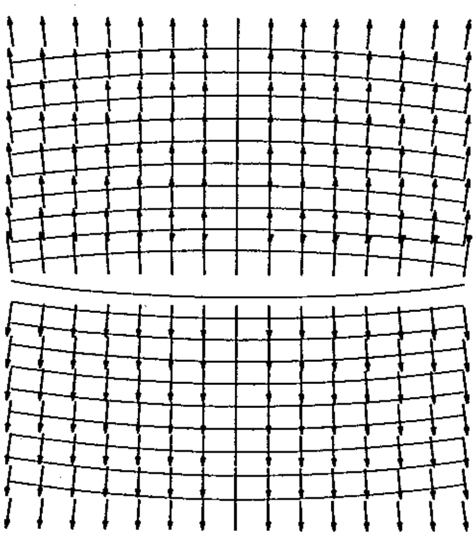
-ContourGraphics-

```
PlotGradientField[V[x,y], {x,-.1,.1}, {y,-.1,.1}]
```



-Graphics-

```
Show[%,%%]
```

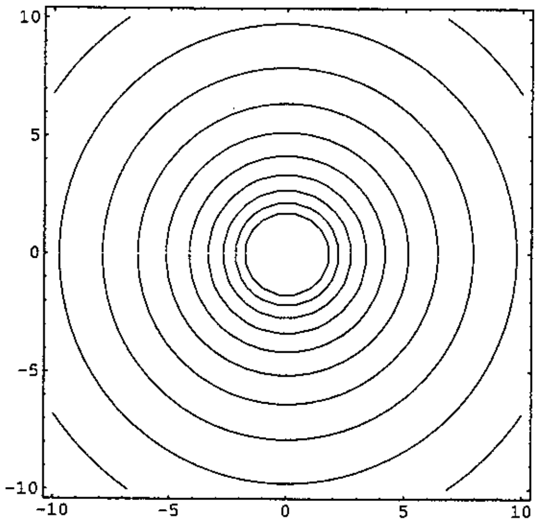


-Graphics-

■ ~~x and y on the big compared to L (looks more like a line charge)~~

In[24]:=

```
ContourPlot[V[x,y], {x,-10,10}, {y,-10,10}, ContourShading->False,
            PlotPoints->30]
```

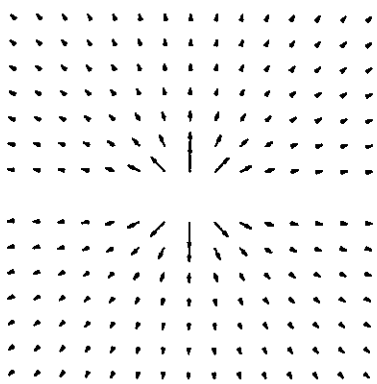


Out[24]=

-ContourGraphics-

In[26]:=

```
PlotGradientField[V[x,y], {x,-10,10}, {y,-10,10}]
```

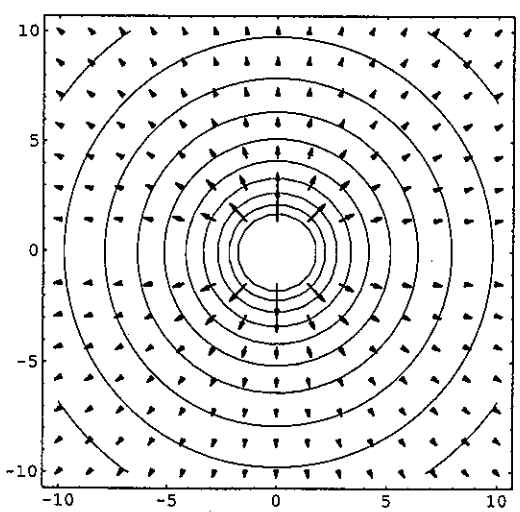


Out[26]=

-Graphics-

In[27]:=

```
Show[%24,%26]
```



Out[27]=

-Graphics-

Two oppositely charged strips

In[28]:=

```
V2[x_,y_] = V[x,y] - V[x,y+s]
```

Principle of superposition!

Out[28]=

$$2\, y \left(\mathrm{ArcTan}\left[\frac{\frac{1}{2} - x}{y}\right] + \mathrm{ArcTan}\left[\frac{\frac{1}{2} + x}{y}\right]\right) - 2\,(s + y)\left(\mathrm{ArcTan}\left[\frac{\frac{1}{2} - x}{s + y}\right] + \mathrm{ArcTan}\left[\frac{\frac{1}{2} + x}{s + y}\right]\right) -$$
$$\left(-\left(\tfrac{1}{2}\right) + x\right) \mathrm{Log}\left[\left(-\left(\tfrac{1}{2}\right) + x\right)^2 + y^2\right] + \left(\tfrac{1}{2} + x\right) \mathrm{Log}\left[\left(\tfrac{1}{2} + x\right)^2 + y^2\right] +$$
$$\left(-\left(\tfrac{1}{2}\right) + x\right) \mathrm{Log}\left[\left(-\left(\tfrac{1}{2}\right) + x\right)^2 + (s + y)^2\right] - \left(\tfrac{1}{2} + x\right) \mathrm{Log}\left[\left(\tfrac{1}{2} + x\right)^2 + (s + y)^2\right]$$

spacing s on the order of the thickness L

In[30]:=

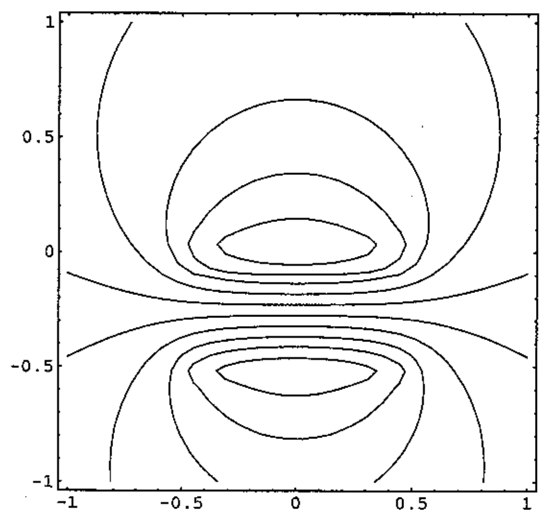
```
s=.5
```

Out[30]=

```
0.5
```

In[31]:=

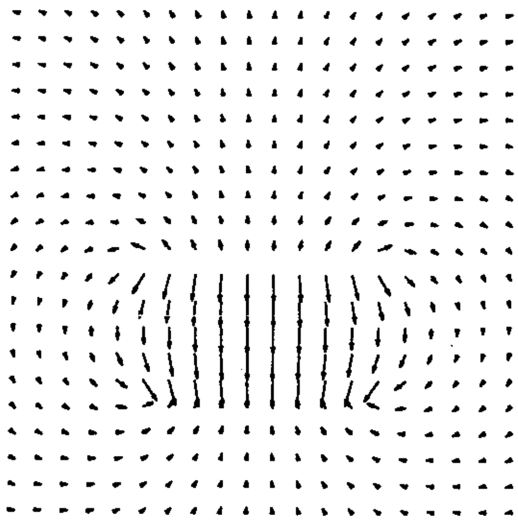
```
ContourPlot[V2[x,y], {x,-1,1}, {y,-1,1}, ContourShading->False,
            PlotPoints->30]
```



Out[31]=

```
-ContourGraphics-
```

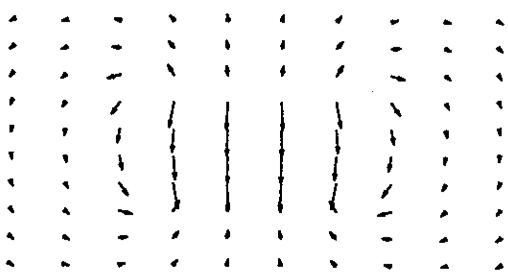
```
PlotGradientField[V2[x,y], {x,-1,1}, {y,-1,1},
                  PlotPoints->20]
```



Out[32]=

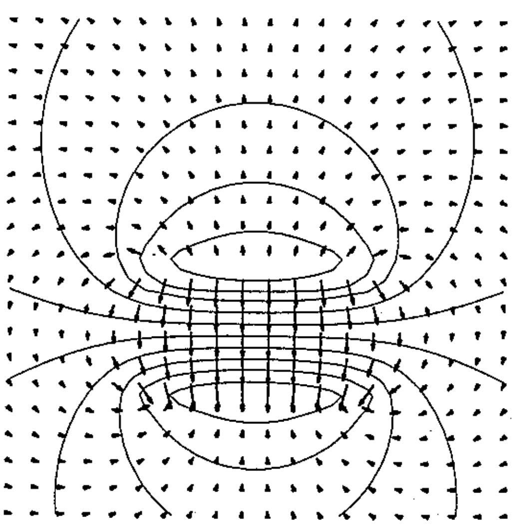
-Graphics-

```
PlotGradientField[V2[x,y], {x,-1,1}, {y,-.75,.25},
                  PlotPoints->10]
```



-Graphics-

```
Show[%,%%]
(* Field somewhat uniform inside.  Smaller fringing fields outside*)
```



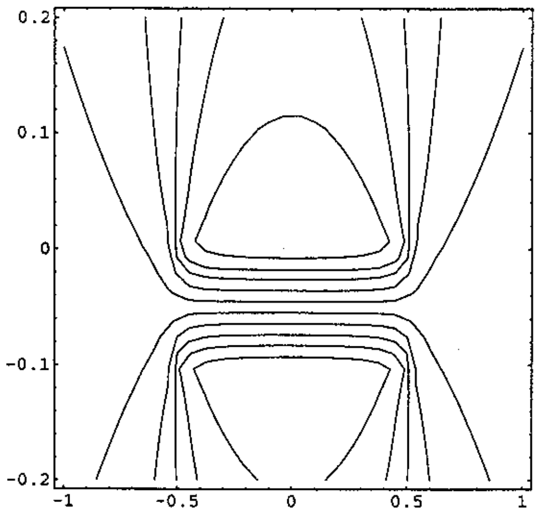
-Graphics-

Spacing s small compared the width L

```
s=.1
```

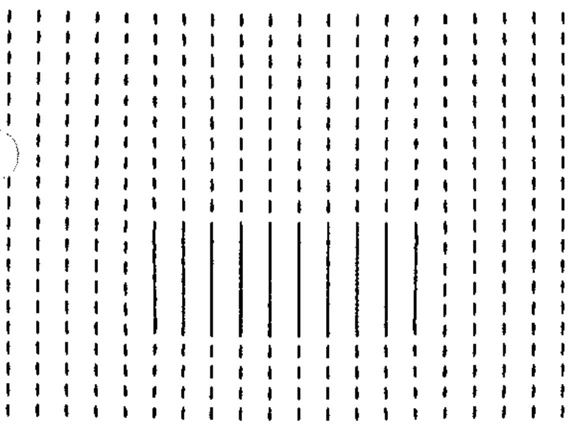
```
0.1
```

```
ContourPlot[V2[x,y],{x,-1,1}, {y,-.2,.2}, ContourShading->False,
          PlotPoints->30]
```



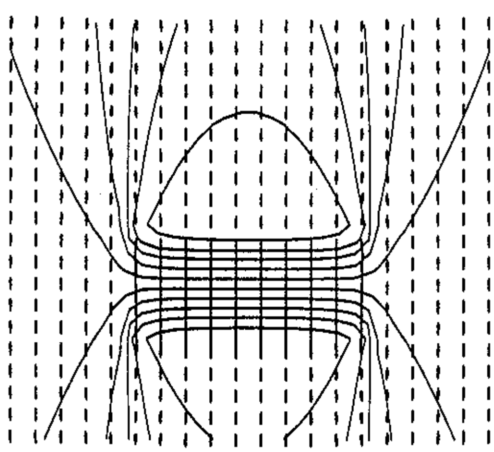

```
-ContourGraphics-
```

```
PlotGradientField[V2[x,y], {x,-1,1}, {y,-.2,.2},
               PlotPoints->20]
```



```
-Graphics-
```

```
Show[%19,%20]
(* Field very uniform inside, very little fringing fields *)
```



```
-Graphics-
```

■ Spacing s very large compared to width L

In[33]:=

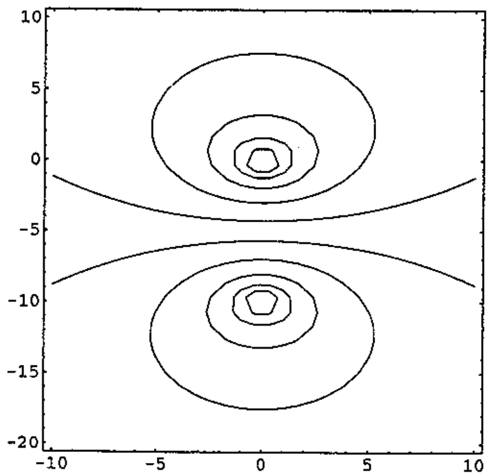
```
s=10
```

Out[33]=

```
10
```

In[34]:=

```
ContourPlot[V2[x,y],{x,-10,10}, {y,-20,10}, ContourShading->False,
        PlotPoints->30]
```

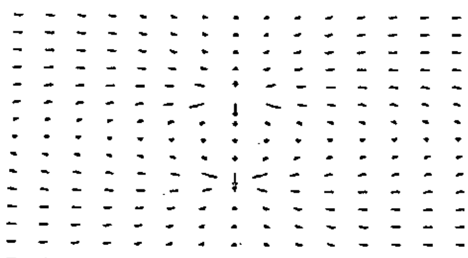


Out[34]=

```
-ContourGraphics-
```

In[36]:=

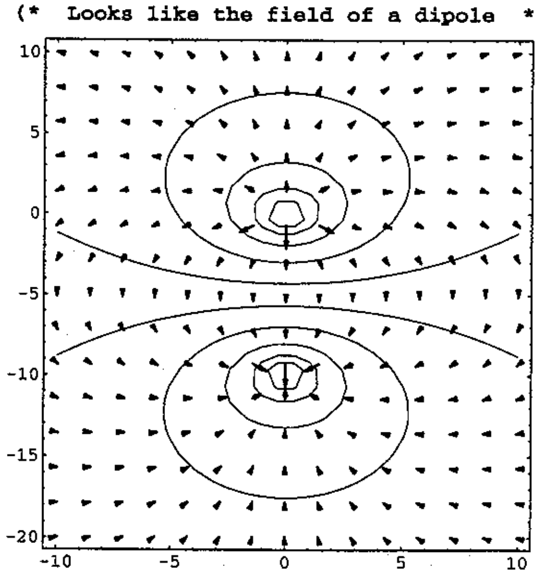
```
PlotGradientField[V2[x,y],{x,-10,10}, {y,-20,10}]
```



Out[36]=

```
-Graphics-
```

```
Show[%34,%36]

(*  Looks like the field of a dipole  *)
```

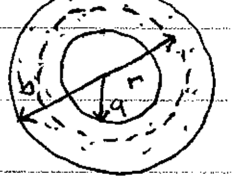


Out[37]=

```
-Graphics-
```

3) Boundary condition at a surface charge

(i) Spherical shell: $\vec{E} = E(r)\,\hat{r}$ by symmetry



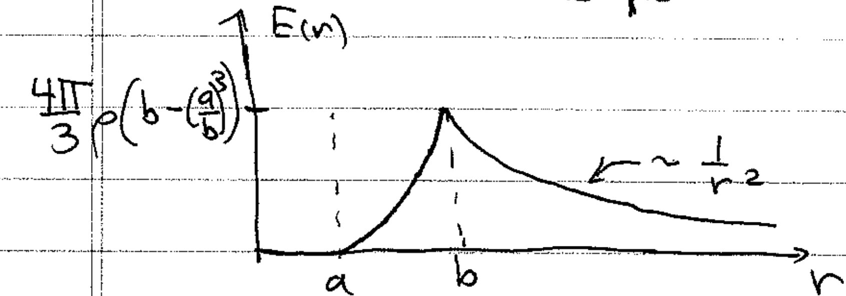
$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E(r) = 4\pi Q_{enc}(r)$$

$$\Rightarrow E(r) = \frac{Q_{enc}(r)}{r^2}$$

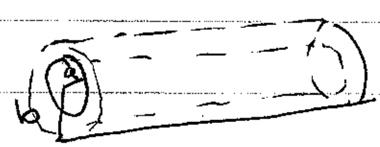
The enclosed charge

$$Q_{enc} = \int_0^r \rho(r')\,4\pi r'^2 dr' = \begin{cases} 0 & r<a \\ \rho\int_a^r 4\pi r'^2 dr' = \frac{4\pi}{3}\rho\,(r^3-a^3) & a<r<b \\ \frac{4\pi}{3}\rho\,(b^3-a^3) & r>b \end{cases}$$

$$\Rightarrow \quad E(r) = \begin{cases} 0 & r<a \\ \frac{4\pi}{3}\rho\left(r-\frac{a^3}{r^2}\right) & a<r<b \\ \frac{4\pi}{3}\frac{\rho}{r^2}\,(b^3-a^3) & r>b \end{cases}$$



(ii) Cylindrical Shell:

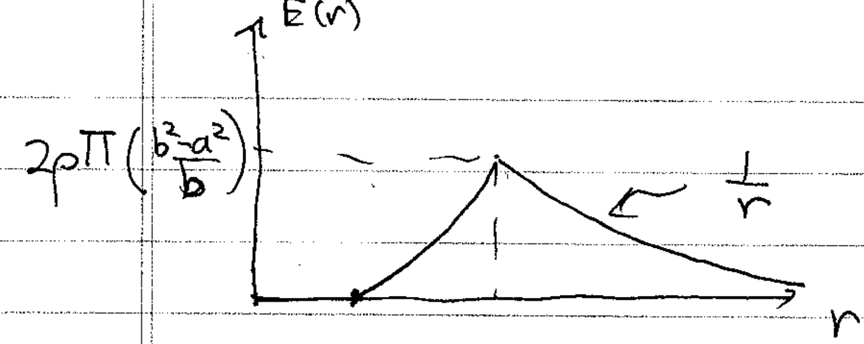


Again $\oint \vec{E}\cdot d\vec{a} = (2\pi r L)\,E(r) = 4\pi Q_{enc}(r) \Rightarrow E(r) = \frac{2Q_{enc}(r)}{Lr}$

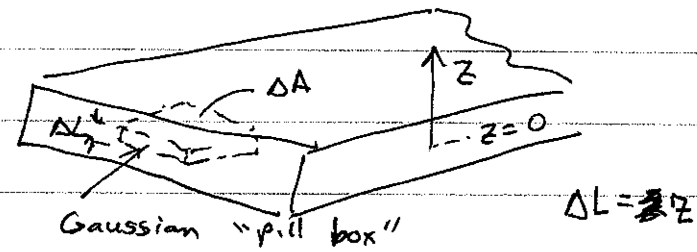
$$Q_{enc}(r) = \int_0^r \rho(r')\,2\pi r' L\,dr' = \begin{cases} 0 & r<a \\ 2\pi L\rho\int_a^r r'dr' = \rho L\pi(r^2-a^2) & a<r<b \\ \rho L\pi(b^2-a^2) & r>b \end{cases}$$

$$\Rightarrow E(r) = \begin{cases} 0 \\ 2\rho\pi\left(r-\frac{a^2}{r}\right) & a<r<b \\ 2\rho\pi\left(\frac{b^2-a^2}{r}\right) & r>b \end{cases}$$

(ii) Cylindrical Shell (sketch)



(iii) Infinite slab

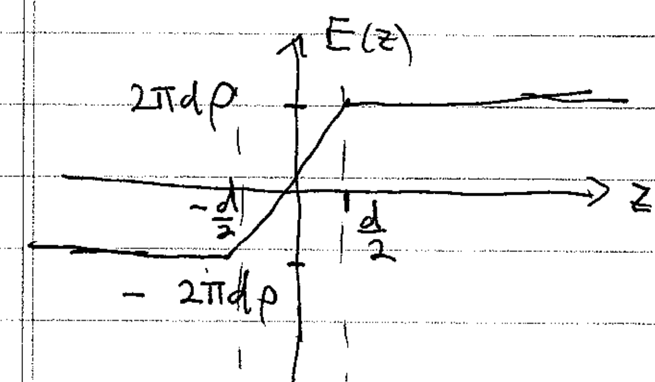


From symmetry, field is uniform, $\perp$ to slab, points away from slab

$$\Rightarrow \oint \vec{E}\cdot d\vec{a} = 2\Delta A\, E(z) = 4\pi\, Q_{enc}(z)$$

$$\Rightarrow |E(z)| = 2\pi \frac{Q_{enc}(z)}{\Delta A} \qquad Q_{enc}(z) = \begin{cases} \rho\,(z)(\Delta A) & |z|<d \\ \rho\,(d)(\Delta A) & |z|>d \end{cases}$$

$$\Rightarrow E(z) = \begin{cases} -2\pi\rho(d) & z<-d \\ +2\pi\rho(z) & -d<z<d \\ 2\pi\rho(d) & z>d \end{cases}$$

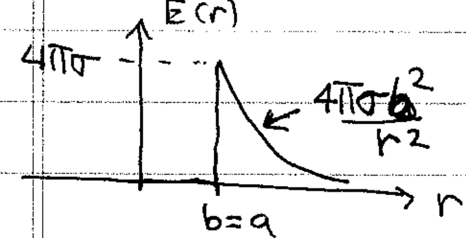


Now take limit as thickness goes to zero, while charge concentrated on surface goes to a constant

(i) $b \equiv a+\delta \qquad \delta \to 0 \qquad \rho\delta \to \sigma$

$$E(r=b) = \frac{4\pi}{3}\frac{\rho}{a^2}\left((a+\delta)^3 - a^3\right) = \frac{4\pi}{3a^2}\rho a^3\left[\left(1+\frac{\delta}{a}\right)^3 - 1\right]$$

$$\lim_{\delta\to 0} E(r=b) \approx \frac{4\pi}{3}\rho a\left[1+\frac{3\delta}{a} - 1\right] = 4\pi\rho\delta \to 4\pi\sigma$$

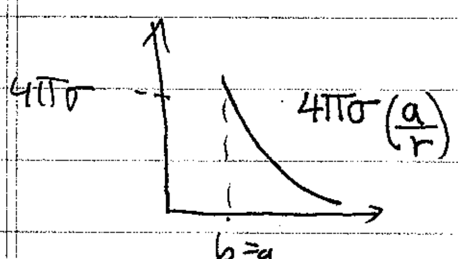


Discontinuity $4\pi\sigma$

(ii) $b \equiv a+\delta \qquad \delta \to 0 \qquad \rho\delta \to \sigma$

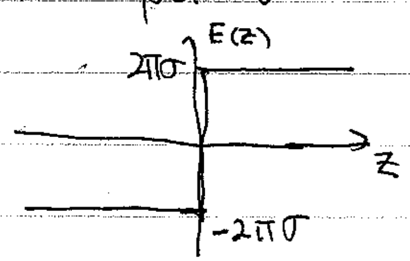
$$E(\underset{=a}{r=b}) = \frac{2\rho\pi}{a}\left[(a+\delta)^2 - a^2\right] = 2\pi\rho a\left[\left(1+\frac{\delta}{a}\right)^2 - 1\right]$$

$$\lim_{\delta\to 0} \cong 2\pi\rho a\left[1+\frac{2\delta}{a} - 1\right] = 4\pi\rho\delta \to 4\pi\sigma$$



Discontinuity $4\pi\sigma$

(iii) $\lim \rho d \to \sigma \qquad d \to 0$



Discontinuity $4\pi\sigma$