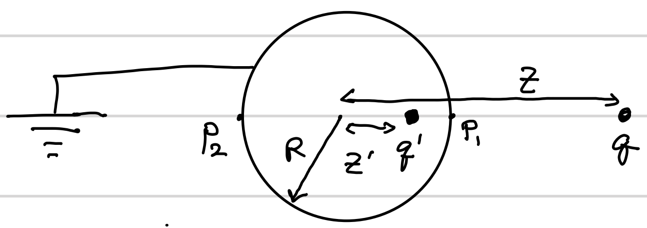
Physics 511: Electrodynamics
Problem Set #3: Solutions

Problem 1: Solution to Poisson's Equation outside a conducting sphere

(a) A point charge is placed a distance z away from the center of a grounded conducting sphere. We take this on the z-axis. To find the potential outside the sphere we use the method of images. The goal is to find a point charge inside the sphere so that its potential, plus the physical charge satisfy the (Dirichlet) boundary conditions. There's guarantee that this will work, nor is there any recipe for finding the image charges. We just have to make an educated guess and hope it works. Luckily in this case it does!

By symmetry we place the image charge also on the z-axis.



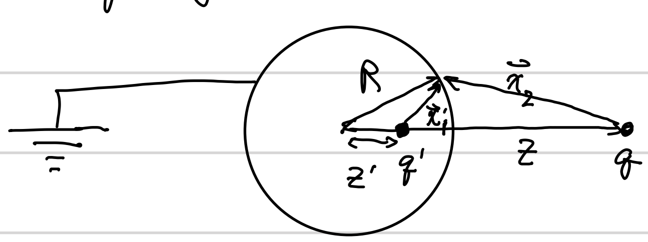
There are two unknowns, q' and z'! Thus we need to equations. We require the electrostatic potential to be zero everywhere. We choose two points to ensure these constraints and then double check that it holds everywhere. The points I choose are P_1 + P_2 as these are simplest, by symmetry.

$$\phi(P_1) = \frac{q}{z-R} + \frac{q'}{R-z'} = 0 \quad \text{and} \quad \phi(P_2) = \frac{q}{z+R} + \frac{q'}{z'+R} = 0$$

$$\Rightarrow q' = -q\,\frac{R-z'}{z-R} = -q\,\frac{z'+R}{z+R} \Rightarrow (z+R)(R-z') = (z-R)(z'+R)$$

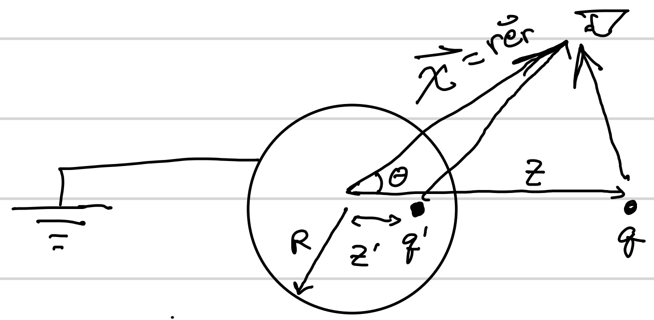
Solving $\Rightarrow$ $\boxed{z' = \frac{R^2}{z}, \quad q' = -q\,\frac{R}{z}}$ Note: As $z/R \to \infty$ $q' \to 0$ as expected

It is a matter of algebra to show that with this image charge



$$\frac{q}{|\vec{x}_1|} + \frac{q'}{|\vec{x}_2|} = 0$$ everywhere on the surface of the sphere.

Now the potential at an arbitrary point outside the sphere



$$\phi(\vec{x}) = \frac{q}{|\vec{x}-z\vec{e}_z|} + \frac{q'}{|\vec{x}-z'\vec{e}_z|} = \frac{q}{\sqrt{\vec{x}\cdot\vec{x}+z^2-2z\vec{x}\cdot\vec{e}_z}} + \frac{q'}{\sqrt{\vec{x}\cdot\vec{x}+z'^2-2z'\vec{x}\cdot\vec{e}_z}}$$

$$= \frac{q}{\sqrt{r^2+z^2-2zr\cos\theta}} + \frac{-q\,R/z}{\sqrt{r^2+\frac{R^4}{z^2}-2r\frac{R^2}{z}\cos\theta}}$$

$$\Rightarrow \boxed{\phi(\vec{x}) = \frac{q}{\sqrt{r^2+z^2-2rz\cos\theta}} - \frac{q}{\sqrt{\left(\frac{rz}{R}\right)^2+R^2-2rz\cos\theta}} \quad \text{for } r \geq R}$$

Note: As required, $\phi(\vec{x}=R\vec{e}_r) = 0$ (grounded sphere)

(b) According to the boundary condition, the surface charge density induced on the conducting sphere caused by the nearby point charge is

$$\sigma(\theta) = \frac{1}{4\pi}\vec{e}_r\cdot\vec{E}(R,\theta) = \frac{1}{4\pi}\left(-\vec{e}_r\cdot\vec{\nabla}\phi\Big|_{r=R}\right) = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_{r=R}$$

Plugging in $$\frac{\partial\phi}{\partial r}\Big|_{r=R} = q\left[\frac{-r+z\cos\theta}{(r^2+z^2-2rz\cos\theta)^{3/2}} - \frac{-\frac{rz^2}{R^2}+z\cos\theta}{\left(\left(\frac{rz}{R}\right)^2+R^2-2rz\cos\theta\right)^{3/2}}\right]_{r=R}$$

$$= \frac{q}{R}\,\frac{z^2-R^2}{(R^2+z^2-2Rz\cos\theta)^{3/2}}$$

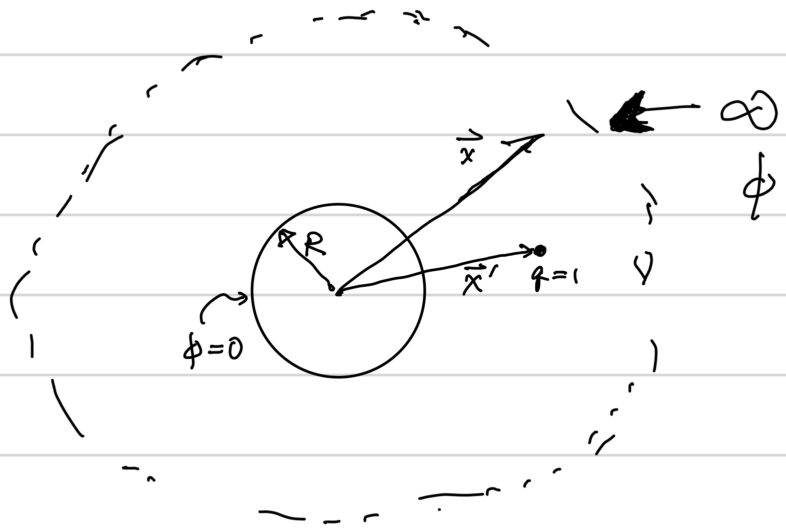
$$\Rightarrow \boxed{\sigma(\theta) = -\frac{q}{4\pi R^2}\,\frac{\left(\frac{z}{R}\right)^2-1}{\left[\left(\frac{z}{R}\right)^2+1-2\frac{z}{R}\cos\theta\right]^{3/2}}}$$

Note $$\int d\Omega\,\sigma(\theta) = \int d(\cos\theta)\,2\pi R^2\,\sigma(\theta) = \frac{q}{2}(a^2-1)\underbrace{\int_{-1}^{1} du\,\frac{1}{[a^2+1-2au]^{3/2}}}_{-\frac{1}{a}(a^2+1-2au)^{-1/2}\Big|_{-1}^{1}}$$ where $u = \cos\theta$, $a = z/R$

$$= +\frac{q}{2}(a^2-1)\,a\left[\frac{1}{a+1} - \frac{1}{a-1}\right]$$

$$= \frac{-q}{a} = -q\frac{R}{z} = q'$$ as expected by Gauss's Law.

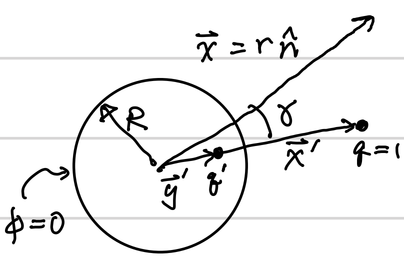
(c) With Dirichelet boundary conditions, the Green's function $G(\vec{x}-\vec{x}') = \phi_{\vec{x}'}(\vec{x})$, the potential ~~potential~~ at position $\vec{x}$ (within the volume of interest) given a unit point charge at $\vec{x}'$ with $\phi|_S = G(\vec{x}-\vec{x}')|_{\vec{x}=S} = 0$ everywhere on the surface. Here, the bounding surface is the conducting sphere and the surface at $|\vec{x}|=\infty$.



But we just solved this problem in part (a), except for the charge fixed on the z-axis. Now it can be at an arbitrary point $\vec{x}'$ outside the conducting sphere. W

Now we place the image charge along $\vec{x}' = r'\hat{n}'$, $q' = -q\frac{R}{|\vec{x}'|} = -\frac{R}{r'}$

$|\vec{y}'| = \frac{R^2}{|\vec{x}'|} \Rightarrow \vec{y} = \frac{R^2}{r'}\hat{n}'$



$$\boxed{G(\vec{x}-\vec{x}') = \phi_{\vec{x}'}(\vec{x}) = \frac{q}{|\vec{x}-\vec{x}'|} + \frac{q'}{|\vec{x}-\vec{y}'|} = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R/r'}{|\vec{x}-\frac{R^2}{r'}\hat{n}'|} = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{r'|\vec{x}-\frac{R^2}{r'^2}\vec{x}'|}}$$

(d) According to Green's Theorem, with Dirichelet boundary conditions

$$\Phi(\vec{x}) = \int d^3x'\, \rho(\vec{x}')\, G(\vec{x}-\vec{x}') - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\, \vec{\nabla}' G(\vec{x}-\vec{x}')\cdot\hat{n}_s'\, da'$$

We seek the solution to Laplace's Eq. $\nabla^2\Phi = 0$ in the volume of interest $\Rightarrow \rho(\vec{x}) = 0$ in volume

$\Rightarrow$ we must find $\vec{\nabla}' G(\vec{x}-\vec{x}')\cdot\hat{n}_s'$ where here $\hat{n}_s'$ is the normal to surface away from volume

$\Rightarrow \hat{n}_s' = -\hat{n}'$ defined in part (c) $\quad \hat{n}_s'\cdot\vec{\nabla}G(\vec{x}-\vec{x}')\Big|_S = -\frac{\partial G}{\partial r'}\Big|_{r=R}$

$$\boxed{G(r\hat{r} - r'\hat{r}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R}{r'\sqrt{r^2+\frac{R^4}{r'^2}-2\frac{R^2 r}{r'}\cos\gamma}} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{R^2}+R^2-2rr'\cos\gamma}} \qquad \cos\gamma = \hat{r}\cdot\hat{r}'}$$

After some algebra: $\left.\frac{\partial G}{\partial r'}\right|_{r=R} = -\frac{r^2-R^2}{R(r^2+R^2-2Rr\cos\gamma)^{3/2}}$, $da' = R^2 d\Omega'$

Thus, by Green's theorem, with Dirichelet boundary condition $\Phi(R,\theta,\phi)$

$$\boxed{\begin{aligned} \Phi(r,\theta,\phi) &= \frac{1}{4\pi}\int d\Omega' \, \Phi(R,\theta',\phi') \frac{R(r^2-R^2)}{(r^2+R^2-2rR\cos\gamma)^{3/2}} \\ \cos\gamma = \hat{r}\cdot\hat{r}' &= \cos\theta\cos\theta' + \sin\theta\sin\theta' \underbrace{(\cos\phi\cos\phi' + \sin\phi\sin\phi')}_{\cos(\phi-\phi')} \end{aligned}}$$

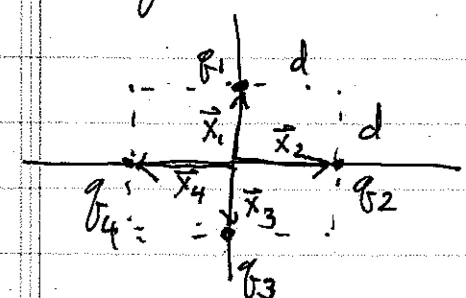
Problem 2

(a) For a discrete set of charges, the charge density is $\rho(\vec{x}) = \sum_\alpha q_\alpha \, \delta^{(3)}(\vec{x}-\vec{x}_\alpha)$, where α labels the particle

$$\Rightarrow Q^{(\ell)}_{i,j,\dots,\ell} \cong \int d^3x \, \rho(\vec{x}) \, [x_i x_j \cdots x_\ell]^{(\ell)} = \sum_\alpha q_\alpha \, [x_i(\alpha) x_j(\alpha) \cdots x_\ell(\alpha)]^{(\ell)}$$

i^{th} coordinate of α^{th} particle

Configuration (i)



$q_1 = 3q$, $q_2 = q_3 = -2q$, $q_4 = q$

$\vec{x}_1 = d\hat{z}$, $\vec{x}_2 = d\hat{x}$

$\vec{x}_3 = -d\hat{x}$, $\vec{x}_4 = -d\hat{z}$

Monopole moment: $q_{net} = \sum_\alpha q_\alpha = 0$

Dipole moment: $\vec{p} = \sum_\alpha \vec{x}_\alpha q_\alpha = q_1\vec{x}_1 + q_2\vec{x}_2 + q_3\vec{x}_3 + q_4\vec{x}_4$

$$\Rightarrow \vec{p} = 3qd\hat{z} - 2qd\hat{x} - 2q(-d\hat{x}) + q(-d\hat{z}) = \boxed{2qd\hat{z}}$$

Quadrapole: $Q_{ij} = \sum_\alpha [3x_i(\alpha)x_j(\alpha) - r_\alpha^2 \delta_{ij}] \, q_\alpha$

$$\Rightarrow Q_{xx} = \sum_\alpha (3x^2(\alpha) - r_\alpha^2) q_\alpha = \{(3\cdot 0 - d^2)(3q) +$$
$$+ (3\cdot d^2 - d^2)(-2q) + (3\cdot d^2 - d^2)(-2q) + (3\cdot 0 - d^2)q\}$$

$$\Rightarrow \boxed{Q_{xx} = -12qd^2}$$ Next page

$$Q_{yy} = \sum_\alpha (3y_{(\alpha)}^2 - r_\alpha^2)\, q_\alpha = -\sum_\alpha r_\alpha^2 q_\alpha \quad \left(\begin{matrix}\text{Since } y\text{-coord}\\ \text{is zero } \forall \text{ charges}\end{matrix}\right)$$

$$= -d^2 \sum_\alpha q_\alpha \quad (\text{since } r_\alpha^2 = d^2 \ \forall \text{ charges})$$

$$\Rightarrow \boxed{Q_{yy} = 0} \quad (\text{since } \sum q_\alpha = q_{net} = 0)$$

Since $Q_{xx} + Q_{yy} + Q_{zz} = 0 \Rightarrow \boxed{Q_{zz} = -Q_{xx} = 12qd^2}$

Since the x-axis and z-axis are the "principle axes" (think about moment of inertia) Q_{ij} is diagonal:

<u>Check:</u> $Q_{xy} = Q_{yx} = Q_{yz} = Q_{zy} = 0$ since y-coordinate $= 0$

$$Q_{xz} = Q_{zx} = 3\sum_\alpha q_\alpha x_\alpha z_\alpha = 0 \quad \left(\begin{matrix}\text{Since } z\text{-coord} = 0 \text{ when}\\ x \text{ nonzero \& vice versa}\end{matrix}\right)$$

$$\therefore Q_{ij} = \begin{bmatrix} -12qd^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12qd^2 \end{bmatrix} = +12qd^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Spherical moments</u>

$$q_{\ell,m} = \sum_\alpha q_\alpha r_\alpha^\ell Y^*_{\ell,m}(\theta_\alpha, \phi_\alpha)$$

We can use spherical ~~coordinates~~ coordinates of charge position, or express $r^\ell Y_{\ell,m}(\theta,\phi)$ in cartesian coordinates as in Jackson (4.4) – (4.6). Here I'll show the latter

(next page)

(1a) Spherical moments, distribution (i)

Monopole: $q_{0,0} = 0$ (net charge)

Dipole: $q_{1,1} = \sum_\alpha q_\alpha r_\alpha Y^*_{1,1}(\theta_\alpha, \phi_\alpha) = -\sqrt{\frac{3}{8\pi}} \sum_\alpha q_\alpha (x_\alpha + i y_\alpha)^*$

$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y) = 0 = q_{1,-1}$

$q_{1,0} = \sum_\alpha q_\alpha r_\alpha Y^*_{1,0}(\theta_\alpha, \phi_\alpha) = \sqrt{\frac{3}{4\pi}} \sum_\alpha q_\alpha z_\alpha = 0$ $\cancel{=q_{1,0}}$

Quadrapole: $q_{2,2} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,2}(\theta_\alpha, \phi_\alpha) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sum_\alpha q_\alpha \left[(x_\alpha + i y_\alpha)^2\right]^*$

$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \left(3q(0) - 2q(d^2) + q(0) - 2q(d^2)\right)$

$= -qd^2 \sqrt{\frac{15}{2\pi}} = q_{2,-2} = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{xx} - 2iQ_{xy} - Q_{yy})$

$q_{2,1} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,1}(\theta_\alpha, \phi_\alpha) = -\sqrt{\frac{15}{8\pi}} \sum_\alpha q_\alpha z_\alpha (x_\alpha - i y_\alpha)^*$

$= 0$ since either $x, y,$ or z coordinate is zero

$= -q_{2,-1}$

$q_{2,0} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,0}(\theta_\alpha, \phi_\alpha) = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \sum_\alpha q_\alpha (3z_\alpha^2 - r_\alpha^2)$

$= \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{zz} = \sqrt{\frac{5}{4\pi}} 6qd^2$

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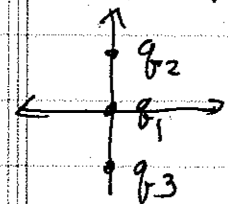
(1a)

For configuration (i)



$q_1 = -q \qquad q_2 = q \qquad q_3 = q$

$\vec{x}_1 = \vec{0} \qquad \vec{x}_2 = d\hat{z} \qquad \vec{x}_2 = -d\hat{z}$

$$\Rightarrow \boxed{q_{net} = q} \qquad \boxed{\vec{p} = 0} \quad \text{(average position)}$$

- $Q_{xy} = Q_{yz} = Q_{xz} = 0$
- $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$

} since distribution is symmetric about z-axis

$$Q_{zz} = \sum_\alpha q_\alpha (3z_\alpha^2 - r_\alpha^2) = q(3d^2 - d^2) + q(3(-d)^2 - d^2) = \boxed{4qd^2}$$

$$Q_{ij} = 4qd^2 \begin{bmatrix} -\frac{1}{2} & & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ "Quadrapole moment"

In spherical moments

$$\boxed{q_{0,0} = \frac{q}{\sqrt{4\pi}}, \quad q_{1,m} = 0, \quad q_{22} = q_{2,1} = 0, \quad q_{2,0} = 2qd^2 \sqrt{\frac{5}{4\pi}}}$$

(1b) $\Phi(\vec{x}) \cong \dfrac{q_{net}}{r} + \dfrac{\vec{x}\cdot\vec{p}}{r^3} + \dfrac{\vec{x}\cdot\overleftrightarrow{Q}\cdot\vec{x}}{2\,r^5}$ (up to order $\frac{d^3}{r^3}$)

Config. (i): $\Phi(\vec{x}) \cong \dfrac{z P_z}{r^3} + \dfrac{Q_{xx}}{2}\dfrac{x^2}{r^5} + \dfrac{Q_{yy}}{2}\dfrac{y^2}{r^5}$

$$\boxed{\Phi(\vec{x}) \cong 2qd\frac{z}{r^3} - 6qd^2\left(\frac{x^2 - z^2}{r^5}\right)}$$

Config (ii): $\Phi(\vec{x}) \cong \dfrac{q_{net}}{r} + \dfrac{Q_{zz}}{2}\dfrac{z^2}{r^5} - \dfrac{Q_{zz}}{4}\left(\dfrac{x^2}{r^5} + \dfrac{y^2}{r^5}\right)$

$$\Rightarrow \boxed{\Phi(\vec{x}) \cong \frac{q_{net}}{r} + \frac{Q_{zz}}{4}\frac{3z^2 - r^2}{r^5} = \frac{q}{r} + qd^2\left(\frac{3z^2 - r^2}{r^5}\right)}$$

(c) The exact potential at position $\vec{x}$

$$\Phi(\vec{x}) = \sum_{\alpha} \frac{q_\alpha}{|\vec{x} - \vec{x}_\alpha|} \quad \text{(superposition)}$$

Configuration (i)

$$\Phi(\vec{x}) = \frac{3q}{|\vec{x} - d\hat{z}|} + \frac{-2q}{|\vec{x} - d\hat{x}|} + \frac{-2q}{|\vec{x} + d\hat{x}|} + \frac{q}{|\vec{x} + d\hat{z}|}$$

$$= \frac{3q}{\sqrt{r^2 - 2zd + d^2}} - \frac{2}{\sqrt{r^2 - 2xd + d^2}} - \frac{2}{\sqrt{r^2 + 2xd + d^2}} + \frac{1}{\sqrt{r^2 + 2zd + d^2}}$$

$$= \frac{q}{r}\left\{3\left(1 - \frac{2zd}{r^2} + \frac{d^2}{r^2}\right)^{-1/2} - 2\left(1 - \frac{2xd}{r^2} + \frac{d^2}{r^2}\right)^{-1/2} - 2\left(1 + \frac{2xd}{r^2} + \frac{d^2}{r^2}\right)^{-1/2} + \left(1 + \frac{2zd}{r^2} + \frac{d^2}{r^2}\right)^{-1/2}\right\}$$

Now expand using $(1+\delta)^{-1/2} \simeq 1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2$, $\delta << 1$

Thus, to order $\left(\frac{d}{r}\right)^3$

$$\Phi(\vec{x}) \simeq \frac{q}{r}\left\{3\left(1 + \frac{zd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\left(-\frac{2zd}{r^2}\right)^2\right) - 2\left(1 + \frac{xd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\left(-\frac{2xd}{r^2}\right)^2\right) - 2\left(1 - \frac{xd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\left(\frac{2xd}{r^2}\right)^2\right) + \left(1 - \frac{zd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\left(\frac{2zd}{r^2}\right)^2\right)\right.$$

(Note: I have dropped all terms, order $\left(\frac{d}{r}\right)^4$ or smaller)

$$\Rightarrow \Phi(\vec{x}) \cong 2qd\frac{z}{r^3} - 6qd^2\left(\frac{x^2 - z^2}{r^5}\right) \quad \text{(as before)}$$

(1c) Continued

Configuration (ii)

$$\Phi(\vec{x}) = -\frac{q}{r} + \frac{q}{|\vec{x}-d\hat{z}|} + \frac{q}{|\vec{x}+d\hat{z}|}$$

$$= -\frac{q}{r} + \frac{q}{\sqrt{r^2-2zd+d^2}} + \frac{q}{\sqrt{r^2+2zd+d^2}}$$

$$= \frac{q}{r}\left\{-1 + \left(1-\frac{2zd}{r^2}+\frac{d^2}{r^2}\right)^{-1/2} + \left(1+\frac{2zd}{r^2}+\frac{d^2}{r^2}\right)^{-1/2}\right\}$$

$$\cong \frac{q}{r}\left\{-1 + \left(1-\frac{zd}{r^2}-\frac{d^2}{2r^2}+\frac{3}{8}\left(\frac{-2zd}{r^2}\right)^2\right)\right.$$

$$\left. + \left(1+\frac{zd}{r^2}-\frac{d^2}{2r^2}\right) + \frac{3}{8}\left(\frac{2zd}{r^2}\right)^2\right)$$

$$\Rightarrow \Phi(\vec{x}) \cong \frac{q}{r}\left\{1-\frac{d^2}{r^2}+\frac{3z^2d^2}{r^4}\right\} = \frac{q}{r} + qd^2\left(\frac{3z^2-r^2}{r^5}\right)$$

✓ As before

(d) In order to plot the equipotentials, let us put the potential in dimensionless form

Configuration (i)

$$\Phi(\vec{x}) = \frac{q}{d}\left\{2\left(\frac{z}{d}\right)\left(\frac{d}{r}\right)^3 - 6\left(\frac{x^2}{d^2}-\frac{z^2}{d^2}\right)\left(\frac{d}{r}\right)^5\right\}$$

Configuration (ii)

$$\Phi(\vec{x}) = \frac{q}{d}\left\{\frac{d}{r} + 3\left(\frac{z^2}{d^2}\right)\left(\frac{d}{r}\right)^5 - \left(\frac{d}{r}\right)^3\right\}$$

All plots in units $\frac{q}{d}$, with distances in units d

Multipole Expansions of Discrete Charge Distributions

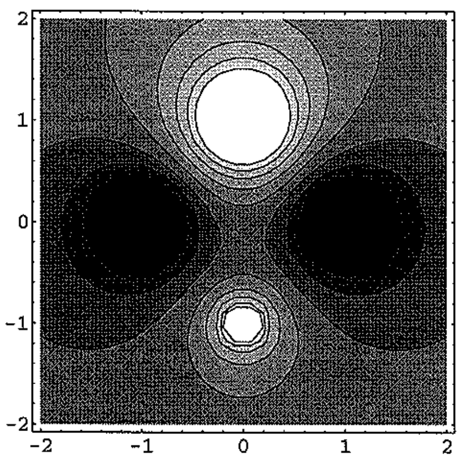
Configuration (i)

Definitions

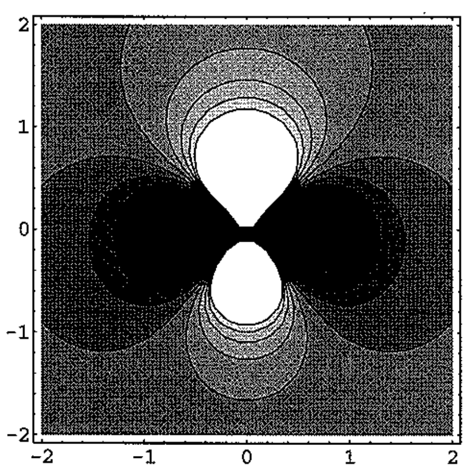
```
Norm[vector_] := Sqrt[vector.vector] (* norm of vector *)

V0[r_,rp_] := 1/Norm[r-rp]
      (* potenial of a unit point charge at rp *)

Vtruei[x_,z_] :=
        Module[{r={x,z}},
              3 V0[r,{0,1}] + V0[r,{0,-1}] -
              2 (V0[r,{1,0}] + V0[r,{-1,0}])]
       (*The exact potential  *)

Vi[x_,z_] := Module[{r=Sqrt[x^2+z^2]},
                     2 z/r^3 - 6 (x^2-z^2)/r^5]
(* Approximate potential including quadrapole correction  *)
```

Close up to the charges

```
(* Exact potential  *)
ContourPlot[Vtruei[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```

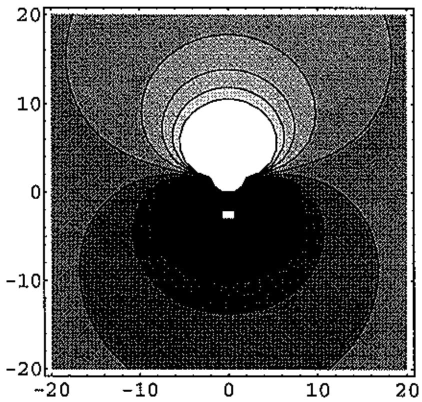


```
(*  Approximate Potential   *)
ContourPlot[Vi[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



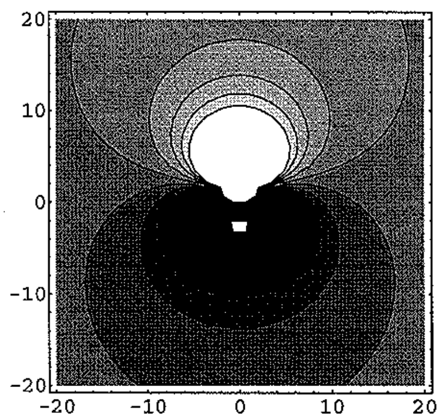
Far away from the charges (Dipole term dominates)

```
(* Exact potential  *)
ContourPlot[Vtruei[x,z],{x,-20,20},{z,-20,20},PlotPoints->30]
```



-ContourGraphics-

```
(*  Approximate Potential   *)
ContourPlot[Vi[x,z],{x,-20,20},{z,-20,20},PlotPoints->30]
```



-ContourGraphics-

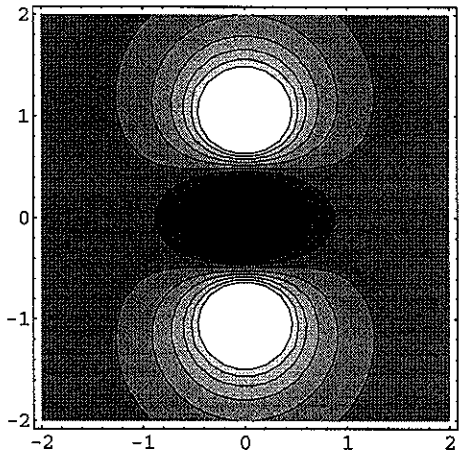
Configuration (ii)

Definitions

```
Vtrueii[x_,z_] :=
        Module[{r={x,z}},
                VO[r,{0,1}] + VO[r,{0,-1}] - VO[r,{0,0}]]
       (*The exact potential  *)

Vii[x_,z_] := Module[{r=Sqrt[x^2+z^2]},
                      1/r + 3 z^2/r^5 - 1/r^3]
(* Approximate potential including quadrapole correction  *)
```

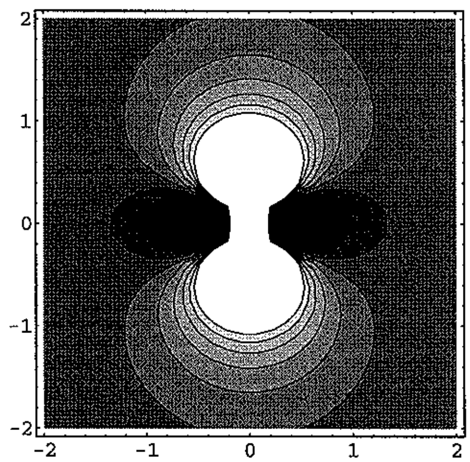
■ Close up to the charges

```
(* Exact potential  *)
ContourPlot[Vtrueii[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



-ContourGraphics-

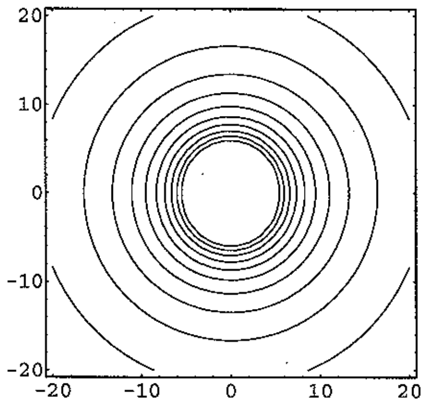
```
(*  Approximate Potential   *)
ContourPlot[Vii[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



-ContourGraphics-

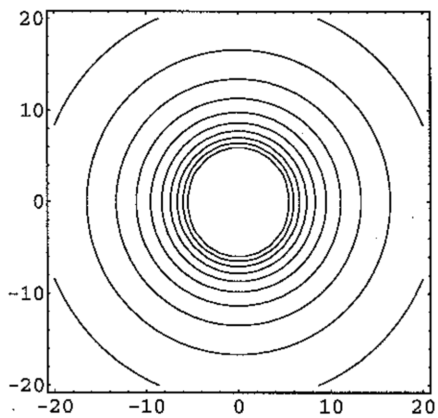
■ Far away from the charges (Monopole term dominates)

```
(* Exact potential  *)
ContourPlot[Vtrueii[x,z],{x,-20,20},{z,-20,20},PlotPoints->30,
            ContourShading->False]
```



-ContourGraphics-

```
(*  Approximate Potential   *)
ContourPlot[Vii[x,z],{x,-20,20},{z,-20,20},PlotPoints->30,
            ContourShading->False]
```



-ContourGraphics-

Problem 3 (Jackson 4.6)

Quadrapole moment $Q = \frac{1}{e} Q_{33}$, in a cylindrically symmetric electric field with $\left.\frac{\partial E_z}{\partial z}\right|_0$ along z axis

(a) Since the field is cylindrically symmetric we can rotate our coordinate axes to lie along the eigenvectors ~~vector~~ of the Q_{ij} tensor

$$\Rightarrow Q_{ij} = \begin{bmatrix} -\frac{1}{2}Q_{33} & 0 & 0 \\ 0 & -\frac{1}{2}Q_{33} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} = eQ \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The energy of quadrapole interaction is given in 4.24, as

$$W = -\frac{1}{6} Q_{ij} \left.\frac{\partial E_j}{\partial x_i}\right|_0 = -\frac{eQ}{6}\left[-\frac{1}{2}\left.\frac{\partial E_x}{\partial x}\right|_0 + \frac{1}{2}\left.\frac{\partial E_y}{\partial y}\right|_0 + \left.\frac{\partial E_z}{\partial z}\right|_0\right)$$

For a cylindrically symmetric field $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$

And since $\vec{\nabla}\cdot\vec{E} = 0$

$$\frac{\partial E_z}{\partial z} = -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right)$$

$$\Rightarrow \boxed{W = \frac{eQ}{4} \left.\frac{\partial E_z}{\partial z}\right|_0}$$

(b) For $Q = 2\times10^{-24}\,\text{cm}^2$, $\frac{W}{h} = 10\,\text{MHz} \Rightarrow W = 6.63\times10^{-20}\,\text{erg} = 6.63\times10^{-27}\,\text{J}$

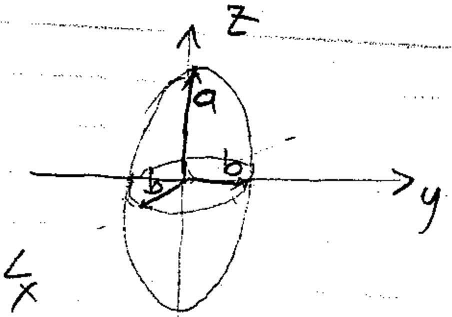
$$\Rightarrow Q = \frac{2\times10^{-24}\,\text{cm}^2}{(0.529\times10^{-8}\,\text{cm})^2}\, a_0^2 = 7.15\times10^{-8}\, a_0^2$$

$$\Rightarrow W = 4.14\times10^{-8}\,\text{eV} = 1.52\times10^{-9}\left(\frac{e^2}{a_0}\right)$$

$\left(\text{Using } \frac{e^2}{2a_0} = \text{Rydberg} = 13.6\,\text{eV}\right)$

$$\therefore 1.52\times10^{-9}\left(\frac{e^2}{a_0}\right) = \frac{-e}{4}\left(7.15\times10^{-8}\, a_0^2\right) \left.\frac{\partial E_z}{\partial z}\right|_0$$

$$\Rightarrow \boxed{\left.\frac{\partial E_z}{\partial z}\right|_0 = -.085\left(\frac{e}{a_0^3}\right)}$$



Charge density: Uniform charge Z_e distributed over the spheroid

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$$

$$\Rightarrow \rho(\vec{x}) = \frac{3Ze}{4\pi ab^2}$$

$$Q = \frac{Q_{zz}}{e} = \frac{1}{e}\int_{\text{spheroid}} (2z^2 - x^2 - y^2)\,\rho(\vec{x})\,d^3x$$

Changing coordinates: $x' = \frac{x}{b}$ $\quad y' = \frac{y}{b}$ $\quad z' = \frac{z}{a}$ $\quad \Rightarrow \underbrace{x'^2 + y'^2 + z'^2 = 1}_{\text{unit sphere}}$

$$\Rightarrow d^3x = \underbrace{\frac{\partial(x,y,z)}{\partial(x',y',z')}}_{\text{Jacobian}} d^3x' = ab^2 d^3x'$$

$$\therefore Q = \frac{3Z}{4\pi}\int_{\text{unit sphere}} \left(2a^2z'^2 - b^2(x'^2+y'^2)\right) d^3x' = \frac{3Z}{4\pi}\int (2a^2\cos^2\theta' - b^2\sin^2\theta')\, r'^2 dV$$

$$= \frac{3Z}{4\pi}\int_0^{2\pi} d\varphi' \int_0^1 r'^4 dr' \int_{-1}^{1} \left[(2a^2+b^2)\cos^2\theta' - b^2\right] d(\cos\theta')$$

$$= \frac{3Z}{4\pi}(2\pi)\left(\frac{1}{5}\right)\left(\frac{2}{3}(2a^2+b^2) - 2b^2\right)$$

$$\boxed{Q = \frac{2}{5}Z(a^2 - b^2)}$$

Example Eu^{153}, $Z = 63$, $Q = 2.5\times 10^{-24}\,\text{cm}^2$, $R = \frac{a+b}{2} = 7\times 10^{-13}\,\text{cm}$

$$\Rightarrow Q = \frac{2}{5}Z(a+b)(a-b) = \frac{4}{5}ZR(a-b)$$

$$\Rightarrow \boxed{\frac{(a-b)}{R} = \frac{5Q}{4ZR^2} = 0.10}$$

Problem 4

Multipole Moments of an azimuthally symmetric charge distribution

Aside

If ρ is symmetric about the z-axis then: ~~$P_x = P_y = 0$~~

- $P_x = P_y = 0$ (average x-position = average y-position = 0)
- $Q_{xy} = Q_{xz} = Q_{yz} = 0$ (principle axes x-y-z)
- $Q_{xx} = Q_{yy}$ (by symmetry) $= -\frac{1}{2} Q_{zz}$ since $\mathrm{Tr}(Q_{ij}) = 0$

$\Rightarrow$ Up to quadrapole term:

$$\Phi(\vec{x}) = \frac{q_{net}}{r} + \frac{z P_z}{r^3} + \frac{1}{2} Q_{zz}\left(\frac{-x^2-y^2}{2} + z^2\right)\frac{1}{r^5}$$

$$= \frac{q_{net}}{r} + P_z \frac{\cos\theta}{r^2} + \frac{1}{4} Q_{zz} \frac{3z^2 - r^2}{r^5}$$

$$= \frac{q_{net}}{r} + P_z \frac{\cos\theta}{r^2} + \frac{1}{4} Q_{zz} \frac{(3\cos^2\theta - 1)}{r^3}$$

$$= \frac{q_{net}}{r} P_0(\cos\theta) + \frac{P_z}{r^2} P_1(\cos\theta) + \frac{Q_{zz}}{2r^3} P_2(\cos\theta)$$

Where $P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3\mu^2 - 1}{2}$ are the Legendre Polynomials

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This also follows from the spherical multipoles

$$q_{\ell,m} \equiv \int d^3x\, \rho(\vec{x})\, r^\ell\, Y^*_{\ell,m}(\theta,\varphi) \quad : \quad \text{If } \rho \text{ } \underline{\text{independent of } \phi}$$

then $q_{\ell,m} = 0$ if $m \neq 0$

$$\Rightarrow \Phi(\vec{x}) = \sum_\ell \frac{4\pi}{2\ell+1}\, q_{\ell,0}\, \frac{Y_{\ell,0}(\theta)}{r^{\ell+1}}$$

$$Y_{\ell,0}(\theta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$\Rightarrow \Phi(\vec{x}) = \sum_\ell Q^{(\ell)}\, \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$$

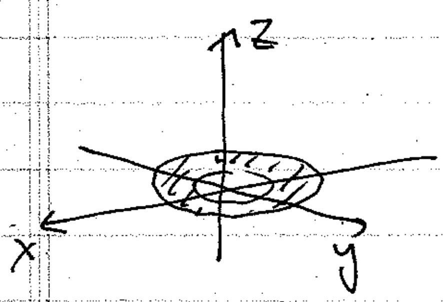
where $Q^{(\ell)} = \sqrt{\frac{4\pi}{2\ell+1}} \int d^3x\, \rho(\vec{x})\, r^\ell\, Y^*_{\ell,0}(\theta,\varphi) = \int d^3x\, \rho(\vec{x})\, r^\ell P_\ell(\cos\theta)$

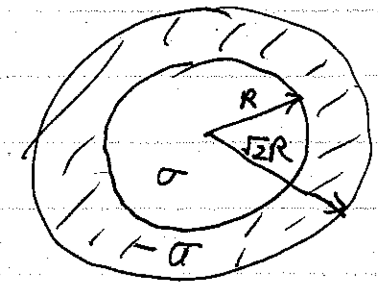
It is easy to show that

$$Q^{(\ell)} = \frac{1}{\ell!} Q^{(\ell)}_{\underbrace{33\cdots3}_{\ell \text{ times}}}$$

(the "z-component" of the ℓ^{th} cartesian tensor)

OK, with that background consider the distribution





$$\boxed{q_{net} = (\pi R^2)\sigma + \left[(\pi 2R^2) - \pi R^2\right](-\sigma) = 0}$$

Problem 4 continued

- $p_x = p_y = 0$ by symmetry.

and $p_z = 0$ since all charge is in x-y plane

- We need $Q_{zz} = \int d^3x\, \rho(\vec{x})\,(3z^2 - r^2) \underset{\uparrow}{=} \int da\, \sigma(r)\,(-r^2)$

all charge at $z=0$

$da = 2\pi r\,dr$ (differential rings)

$$\Rightarrow Q_{zz} = \sigma \int_0^R (-r^2)\, 2\pi r\,dr - \sigma \int_R^{\sqrt{2}R} (-r^2)\, 2\pi r\,dr$$

$$= 2\pi\sigma \left(-\int_0^R r^3\,dr + \int_R^{\sqrt{2}R} r^3\,dr \right)$$

$$= 2\pi\sigma \left(-\frac{r^4}{4}\Big|_0^R + \frac{r^4}{4}\Big|_R^{\sqrt{2}R} \right) = 2\pi\sigma \left(-\frac{R^4}{4} + \frac{4R^4}{4} - \frac{R^4}{4} \right)$$

$$\Rightarrow \boxed{Q_{zz} = \pi\sigma R^4} \quad \text{(units: Charge} \cdot \text{Length}^2\text{)}$$

$$\Rightarrow Q_{ij} \doteq \pi\sigma R^4 \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\Phi(r,\theta) = Q_{zz}\,\frac{P_2(\cos\theta)}{2r^3} = \frac{\pi\sigma R^4}{4}\,\frac{(3\cos^2\theta - 1)}{r^3}}$$

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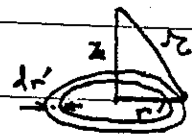
(b) By direct integration, the potential along the z-axis is found by adding up the contribution of rings of charge of radius r' and thickess dr'



$$d\Phi = \frac{\sigma(r')\, da'}{r}, \quad da' = 2\pi r' dr'$$

$$= \frac{2\pi\sigma(r') dr'}{\sqrt{r'^2 + z^2}}$$

$$\Rightarrow \Phi(z) = \int d\Phi = 2\pi\sigma \int \frac{dr'\, \sigma(r')}{\sqrt{r'^2+z^2}}$$

$$= 2\pi\sigma \left[\int_0^R \frac{dr'}{\sqrt{r'^2+z^2}} - \int_R^{\sqrt{2}R} \frac{dr'}{\sqrt{r'^2+z^2}} \right]$$

$$= 2\pi\sigma \left[\sqrt{r'^2+z^2}\Big|_0^R - \sqrt{r'^2+z^2}\Big|_R^{\sqrt{2}R} \right]$$

$$\Rightarrow \Phi(z) = 2\pi\sigma \left(\sqrt{z^2+R^2} - z - \sqrt{z^2+2R^2} + \sqrt{z^2+R^2} \right)$$

$$\Rightarrow \boxed{\Phi(z) = 2\pi\sigma \left(2\sqrt{z^2+R^2} - \sqrt{z^2+2R^2} - z \right)}$$

(c) Since ρ is azimulthally symmetric, we know V is independent of ϕ, and therefore outside the charge distribution

$$\Phi(r,\theta) = \sum_{\ell}{}' \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

for $r > \sqrt{2}R$ we have the b.c. $V \to 0$ and $r \to \infty$

$$\Rightarrow \quad A_\ell = 0 \quad \forall \ell$$

$$\Rightarrow \quad \Phi(r,\theta) = \sum_{\ell}{}' \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

We also have the b.c. $V(r, \theta=0)$ (along z-axis)

$$\Phi(r,\theta=0) = 2\pi\sigma\left(2\sqrt{r^2+R^2} - \sqrt{r^2+2R^2} - r\right)$$

To find the expansion coefficients, B_ℓ, expand the above expression in powers of $\frac{R}{r} \ll 1$

$$\Phi(r,\theta=0) = 2\pi\sigma r\left(2\left(1+\frac{R^2}{r^2}\right)^{1/2} - \left(1+\frac{2R^2}{r^2}\right)^{1/2} - 1\right)$$

$$\approx 2\pi\sigma r\left\{2\left(1+\frac{R^2}{2r^2} - \frac{1}{8}\left(\frac{R^2}{r^2}\right)^2\right) - \left(1+\frac{1}{2}\frac{2R^2}{r^2} - \frac{1}{8}\left(\frac{2R^2}{r^2}\right)^2\right) - 1\right\}$$

here I used $(1+\delta)^n \approx 1 + n\delta + \frac{n(n-1)}{2}\delta^2$ for $\delta < 1$

$$\therefore \Phi(r, \theta=0) \approx 2\pi\sigma r\left(2 + \frac{R^2}{r^2} - \frac{1}{4}\frac{R^4}{r^4} - 1 - \frac{R^2}{r^2} + \frac{1}{2}\frac{R^4}{r^4} - 1\right)$$

$$= 2\pi\sigma r\left(\frac{R^4}{4r^4}\right) = \frac{\pi\sigma R^4}{2}\frac{1}{r^3}$$

The general expansion is

$$\Phi(r,\theta) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow \Phi(r, \theta=0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}}$$

$$= \frac{B_0}{r} + \frac{B_1}{r^2} + \frac{B_2}{r^3} + \cdots$$

$$\Rightarrow B_0 = B_1 = 0 \qquad B_2 = \frac{\pi\sigma R^4}{2}$$

$\therefore$ Up to order $1/r^3$

$$\boxed{\Phi(r,\theta) = \frac{\pi\sigma R^4}{2}\frac{1}{r^3}P_2(\cos\theta)}$$

(as in part (a))