Physics 511

Solutions: Problem Set # 4

Problem 1: Motion of Charged Particles in External Fields

(a) Given a particle with charge q, mass m, moving in external $\vec{E}$ and $\vec{B}$ fields, the basic equation of motion is the Lorentz force law

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad \vec{E}, \vec{B} \text{ time independent}$$

$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q}{mc}\left(\frac{d\vec{v}}{dt}\times\vec{B}\right) = \frac{q^2}{m^2c}\left(\vec{E}\times\vec{B} + \left(\frac{\vec{v}}{c}\times\vec{B}\right)\times\vec{B}\right)$$

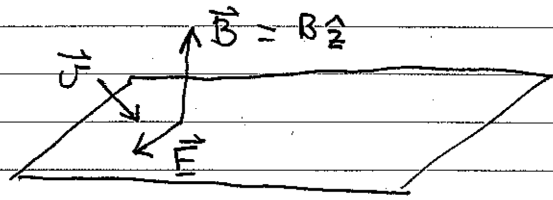
$$\Rightarrow \frac{d^2\vec{v}}{dt^2} = \frac{q^2}{m^2c}\left(-\frac{\vec{v}}{c}(\vec{B}\cdot\vec{B}) + \vec{B}\left(\frac{\vec{v}\cdot\vec{B}}{c}\right) + \vec{E}\times\vec{B}\right)$$

$$\boxed{\frac{d^2\vec{v}}{dt^2} = -\omega_c^2\,\vec{v} + \frac{q^2}{m^2c}(\vec{E}\times\vec{B}) + \frac{q^2}{m^2c^2}(\vec{B}\cdot\vec{v})\vec{B} \qquad \text{where } \omega_c = \frac{qB}{mc} \text{ is the cyclotron frequency}}$$

(b) Let us choose $\vec{B}$ in the $\hat{z}$-direction, and $\vec{E}$ in the x-y plane. If the initial velocity is in the x-y plane the $\vec{v}\cdot\vec{B}=0$ for all time.

The equation of motion then becomes

$$\frac{d^2\vec{v}}{dt^2} = -\omega_c^2\vec{v} + \frac{q^2}{m^2c}(\vec{E}\times\vec{B})$$



$$\vec{v}(0) = v_x\hat{x} + v_y\hat{y}$$

We can solve this equation using Laplace transforms:

$$\vec{V}(s) \equiv \int_0^\infty \vec{v}(t)\, e^{-st}\, dt \equiv \mathcal{L}[\vec{v}(t)]$$

We then have $\mathcal{L}\left[\frac{d^2 v}{dt^2}\right] = s^2 \vec{V}(s) - s\vec{v}(0) - \frac{d\vec{v}}{dt}(0)$

$$\mathcal{L}[1] = \frac{1}{s}$$

Transforming the ODE, we find the algebraic equation

$$s^2 \vec{V}(s) - s\vec{v}(0) - \frac{d\vec{v}}{dt}(0) = -\omega_c^2 \vec{V}(s) + \frac{\vec{F}}{s}$$

(where $\vec{F} \equiv \frac{q^2}{m^2 c}(\vec{E}\times\vec{B})$)

$$\Rightarrow \vec{V}(s) = \frac{s}{s^2+\omega_c^2}\vec{v}(0) + \frac{1}{s^2+\omega_c^2}\frac{d\vec{v}}{dt}(0) + \frac{1}{s(s^2+\omega_c^2)}\vec{F}$$

Taking the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_c^2}\right] = \cos\omega_c t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_c}{s^2+\omega_c^2}\right] = \sin\omega_c t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\omega_c^2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{\omega_c^2}\left(\frac{1}{s} - \frac{s}{s^2+\omega_c^2}\right)\right] = \frac{1}{\omega_c^2}(1-\cos\omega_c t)$$

$$\Rightarrow \vec{v}(t) = \vec{v}(0)\cos\omega_c t + \frac{d\vec{v}}{dt}(0)\frac{\sin\omega_c t}{\omega_c} + (1-\cos\omega_c t)\, c\frac{\vec{E}\times\vec{B}}{B^2}$$

(Having substituted in for $\vec{F}$ and ω_c)

Now from the Lorentz Force Law $\frac{d\vec{v}}{dt}(0) = \frac{q}{m}\left(\vec{E} + \frac{\vec{v}(0)\times\vec{B}}{c}\right)$

$$\Rightarrow \vec{v}(t) = \vec{v}(0)\cos\omega_c t + \frac{q}{m\omega_c}\left(\vec{E} + \frac{\vec{v}(0)}{c}\times\hat{e}_z B\right) + (1-\cos\omega_c t)\frac{c\,\vec{E}\times\vec{B}}{B^2}$$

$$\Rightarrow \boxed{\vec{v}(t) = \left(\vec{v}(0) - c\frac{\vec{E}\times\vec{B}}{B^2}\right)\cos\omega_c t + \left(\vec{v}(0)\times\hat{e}_z + \frac{c\vec{E}}{B}\right)\sin\omega_c t + c\frac{\vec{E}\times\vec{B}}{B^2}}$$

Note that there are <u>two</u> integration constants $v_x(0)$ and $v_y(0)$; $\frac{dv_x}{dt}(0)$ and $\frac{dv_y}{dt}(0)$ are constrained by the Lorentz force Law

We can express this in a more compact form as follows.

- Let $\vec{E}$ be along the x-axis

Breaking the solution into components

$$v_x(t) = \left(v_x(0) - \frac{cE}{B}\right)\cos\omega_c t + v_y(0)\sin\omega_c t + c\frac{E}{B}$$

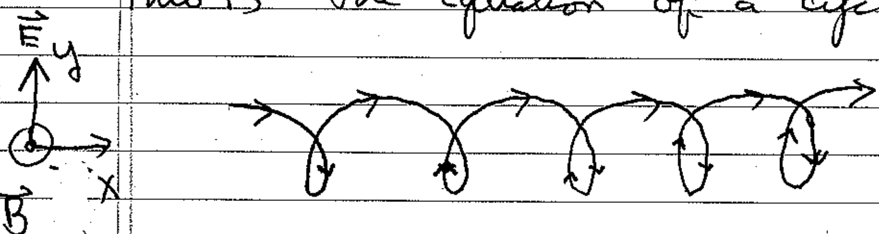
$$v_y(t) = v_y(0)\cos\omega_c t - \left(v_x(0) - \frac{cE}{B}\right)\cos\omega_c t$$

$$\Rightarrow \begin{cases} v_x(t) = u_0\cos(\omega_c t + \phi) + \frac{cE}{B} \\ v_y(t) = -u_0\sin(\omega_c t + \phi) \end{cases}$$

where $u_0 = \sqrt{\left(v_x(0) - \frac{cE}{B}\right)^2 + v_y(0)}$, $\phi = \tan^{-1}\left(\frac{v_y(0)}{v_x(0) - \frac{cE}{B}}\right)$

$$\Rightarrow \boxed{\vec{v}(t) = u_0\cos(\omega_c t + \phi)\hat{e}_x - u_0\sin(\omega_c t + \phi)\hat{e}_y + c\frac{\vec{E}\times\vec{B}}{B^2}}$$

This is the equation of a cycloid



Cyclotron motion with $\vec{E}\times\vec{B}$ drift

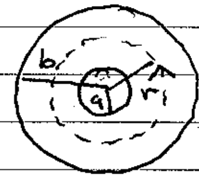
Problem 2: Boundary Condition at a Surface Current

(a) Uniform current densities

(i) Infinite cylindrical shell, current along axis

Symmetry $\Rightarrow$ $\vec{B} = B(r)\hat{\phi}$ (r = cylindrical radius)

In x-y plane
Current out of page



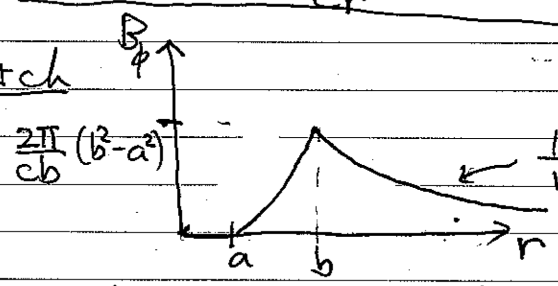
Ampèrian loop at r

$\Rightarrow \oint_c \vec{B}\cdot d\vec{l} = 2\pi r\, B(r) = \frac{4\pi}{c} I_{enc}$

$\Rightarrow B(r) = \frac{2 I_{enc}(r)}{cr}$, $I_{enc}(r) = \begin{cases} 0 & r<a \\ \pi(r^2-a^2)J & a<r<b \\ \pi(b^2-a^2)J & r>b \end{cases}$

$$\Rightarrow \boxed{\vec{B} = \begin{cases} 0 & r<a \\ \frac{2\pi}{cr}(r^2-a^2)J\hat{\phi} & a<r<b \\ \frac{2\pi}{cr}(b^2-a^2)J\hat{\phi} & r>b \end{cases}}$$

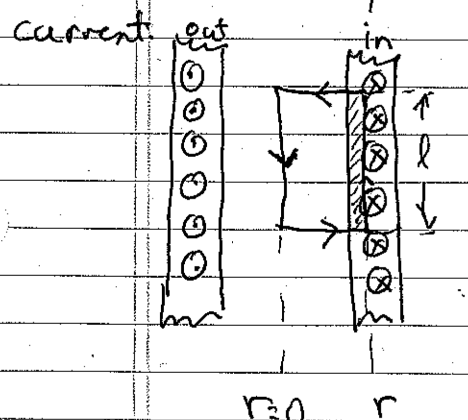
Sketch



(ii) Infinite cylindrical shell, azimuthal current

$\vec{B}$ Symmetry $\Rightarrow$ $\vec{B} = B(r)\hat{z}$ (Like Solenoid)
$B = 0$ outside solenoid
B uniform inside

In x-z plane



$\oint \vec{B}\cdot d\vec{l} = -B(0)\,l + B(r)\,l = -\frac{4\pi}{c} I_{enc}(r)$ ← Right hand rule

If $r>b$, $B(r) = 0$ $I_{enc} = J(b-a)\,l$

$\Rightarrow B(0) = \frac{4\pi}{c} J(b-a)$

(Next Page)

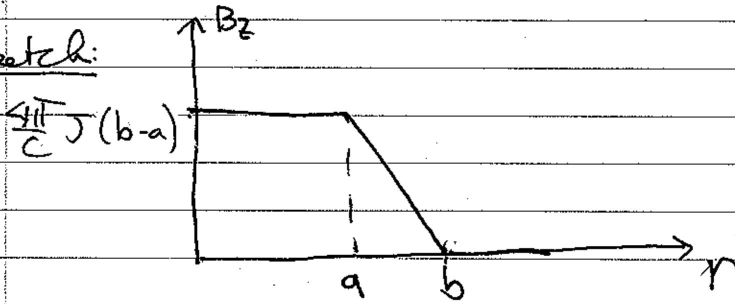
If $r<a \qquad B(r) = B(0) = \frac{4\pi}{c} J(b-a)$

If $a<r<b \qquad I_{enc} = J(r-a)l$

$$\Rightarrow B(r) = B(0) - \frac{4\pi}{c}\frac{I_{enc}}{l} = \frac{4\pi}{c} J(b-r)$$

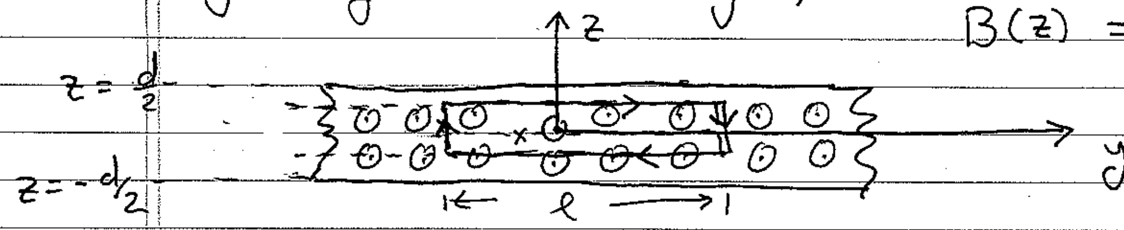
$$\Rightarrow \boxed{\vec{B} = \begin{cases} \frac{4\pi}{c} J(b-a)\,\hat{z} & r\le a \\ \frac{4\pi}{c} J(b-r)\,\hat{z} & a\le r\le b \\ 0 & r\ge b \end{cases}}$$

Sketch:



(iii) Infinite slab, current along x, $\hat{z}$ normal

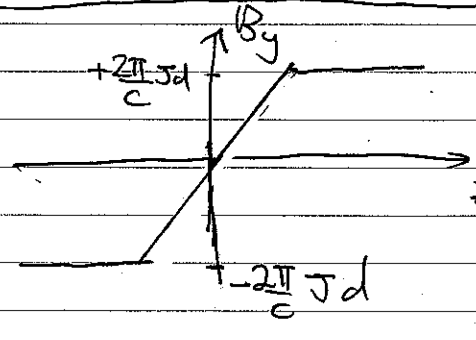
Symmetry: $\vec{B} = B(z)\,\hat{y}$, B constant above or below
$B(z) = -B(-z)$



$$\oint \vec{B}\cdot d\vec{l} = B(z)l + (-B(-z))l = \frac{4\pi}{c} I_{enc} \Rightarrow B(z) = \frac{2\pi}{c}\frac{I_{enc}}{l}$$

$I_{enc} = (2z)\,l\,J \quad |z| \le \frac{d}{2}$, $\quad I_{enc} = d\,l\,J \quad |z| \ge \frac{d}{2}$

$$\Rightarrow \boxed{\vec{B} = \begin{cases} \frac{2\pi}{c} Jd\,\hat{y} & z\ge\frac{d}{2} \\ \frac{4\pi}{c} Jz\,\hat{y} & -\frac{d}{2}\le z\le\frac{d}{2} \\ -\frac{2\pi}{c} Jd\,\hat{y} & z\le -d/2 \end{cases}}$$



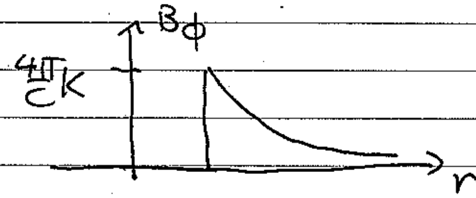
Problem 2b

We now take the limit that the shell shrinks to zero thickness, and all the current is concentrated on the surface

(i) let $b = a + \Delta$, take limit $\Delta \to 0$ $\quad J\Delta \to K$

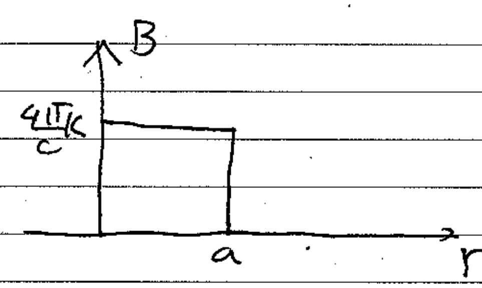
$$\lim_{\Delta \to 0} (b^2 - a^2) J = 2a\Delta J = 2Ka$$

$$\Rightarrow \vec{B} = \begin{cases} 0 & r < a \\ \frac{4\pi}{c}\frac{a}{r} K \hat{\phi} & r > a \end{cases}$$



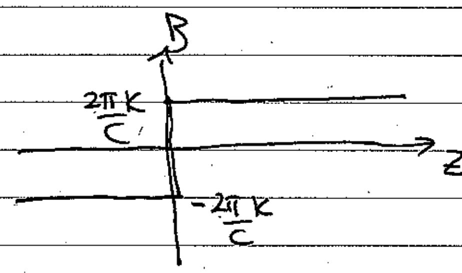
(ii) $\lim_{\Delta \to 0} J(b-a) = K$

$$\vec{B} = \begin{cases} \frac{4\pi}{c} K \hat{z} & r < a \\ 0 & r > a \end{cases}$$



(iii) $\lim_{d \to 0} Jd = K$

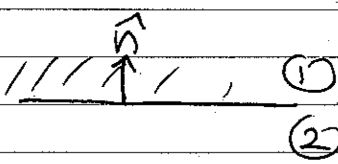
$$\Rightarrow \vec{B} = \begin{cases} \frac{2\pi}{c} K & z > d \\ -\frac{2\pi}{c} K & z < d \end{cases}$$



We thus see a discontinuity of magnitude $\frac{4\pi}{c}K$ in the tangential component of $\vec{B}$

The direction is such that

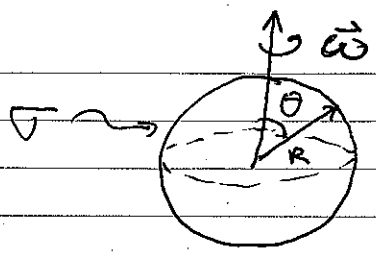
$$\hat{n} \times (\vec{B}_1 - \vec{B}_2) = \frac{4\pi}{c}\vec{K}$$



Note: $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ normal component of $\vec{B}$ is continuous

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

Problem 3 Field of a rotating sphere



Sphere rotates about z-axis with angular velocity $\vec{\omega}$; uniform surface charge density σ covers the sphere

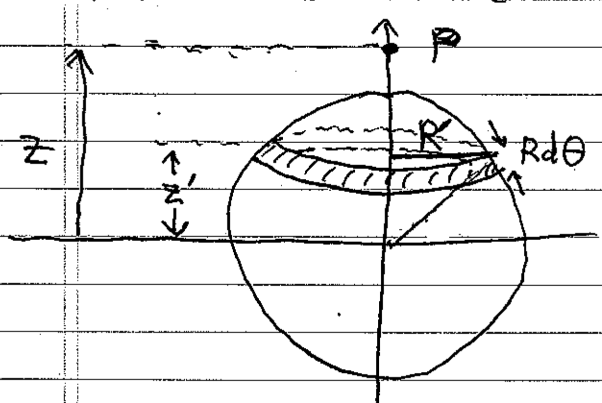
The velocity of an element on the sphere is

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega R(\hat{z} \times \hat{r}) = \omega R \sin\theta \,\hat{\phi}$$

Thus, the surface charge density is

$$\vec{K}(\theta) = \sigma \vec{v} = \sigma \omega R \sin\theta \,\hat{\phi}$$

We seek the magnetic field on the z-axis. Let us break the surface into rings



The ring at (R, θ) is shown

- Radius of ring $R' = R\sin\theta$
- $z' = R\cos\theta$
- Distance from ring to P: $|z - z'|$
- Differential current in ring
$$dI(\theta) = K\,R\,d\theta = \sigma \omega R^2 \sin\theta \, d\theta$$

In class we derived the magnetic field of a current ring radius a, distance Z from axis

$$\vec{B} = \hat{z}\, \frac{2\pi}{c} \frac{a^2 I}{(a^2 + Z^2)^{3/2}}$$

$\Rightarrow$ For the differential ring, the contribution to $\vec{B}$ is

$$d\vec{B} = \hat{z}\, \frac{2\pi}{c} \frac{R^2 \sin^2\theta \;\; dI(\theta)}{\left[(R\sin\theta)^2 + |z - R\sin\theta|^2\right)^{3/2}}$$

$$\Rightarrow d\vec{B} = \hat{z}\,\frac{2\pi}{c}\,\sigma\omega R^4 \,\frac{\sin^3\theta\, d\theta}{(z^2+R^2-2Rz\cos\theta)^{3/2}}$$

Now integrate over surface $\theta: 0 \to \pi$

$$\Rightarrow \vec{B}(z) = \int d\vec{B}(z) = \hat{z}\,\frac{2\pi}{c}\,\sigma\omega R^4 \underbrace{\int_0^{\pi} \frac{\sin^3\theta\, d\theta}{(z^2+R^2-2Rz\cos\theta)^{3/2}}}_{\leftarrow \mathcal{I}}$$

Aside: We must consider two case
(i) $z>R$ (outside sphere) (ii) $z<R$ (inside)

Case (i)

$$\mathcal{I} = \frac{1}{z^3}\int_0^{\pi} \frac{\sin^3\theta\, d\theta}{(1+a^2-2a\cos\theta)^{3/2}} \qquad \text{where } a \equiv \frac{R}{z} < 1$$

This integral can be done by noting

$$\frac{d}{d\theta}(1+a^2-2a\cos\theta)^{1/2} = \frac{-a\sin\theta}{(1+a^2-2a\cos\theta)^{3/2}}$$

and then performing integration by parts

After some
$$\int_0^{\pi} \frac{\sin^3\theta\, d\theta}{(1+a^2-2a\cos\theta)^{3/2}} = \frac{4}{3} \quad (a<1)$$

$$\therefore\ z>R \qquad \mathcal{I} = \frac{4}{3}\frac{1}{z^3}$$

For $R>z$
$$\mathcal{I} = \frac{1}{R^3}\int_0^{\pi} \frac{\sin^3\theta\, d\theta}{(1+a^2-2a\cos\theta)^{3/2}} \qquad a \equiv \frac{z}{R} < 1$$

$$\Rightarrow \quad R>z \qquad \mathcal{I} = \frac{4}{3}\frac{1}{R^3}$$

(Next page)

Thus

$$\vec{B}(z) = \begin{cases} \frac{8\pi R^3}{3}\left(\frac{\sigma\omega R}{c}\right)\frac{1}{z^3}\,\hat{z} & z > R \\ \frac{8\pi\sigma\omega R}{3c}\,\hat{z} & z < R \end{cases}$$

Note that outside the sphere $\vec{B}$ falls off a $\frac{1}{r^3}$

This is an exact result $\Rightarrow$ $\vec{B}$ outside is a dipole

Recall that on the axis of a dipole

$$\vec{B} = \frac{2m}{z^3}\,\hat{z}$$

$$\Rightarrow \vec{m} = \frac{4\pi R^3}{3}\left(\frac{\sigma\omega R}{c}\right)\hat{z}$$

We will prove this in Problem 4

Inside, the magnetic field is independent of z

We will show in the next part that $\vec{B}$ is in fact uniform throughout the inside of the sphere

(Next Page)

b) The vector potential is given by

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x})\, d^3x'}{|\vec{x}-\vec{x}'|} = \frac{1}{c}\int_{\substack{\text{surface}\\\text{of sphere}}} \frac{\vec{K}(\theta')}{|\vec{x}-\vec{x}'|} R^2 d\Omega'$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\sigma\omega R^3}{c}\int \frac{\sin\theta'\hat{\phi}'}{|\vec{x}-\vec{x}'|} d\Omega'$$

Aside:

Remember that $\hat{\phi}'$ is a function of position of current

$$\hat{\phi}' = -\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y}$$

$$\Rightarrow \sin\theta'\hat{\phi}' = -\sin\theta'\sin\phi'\,\hat{x} + \sin\theta'\cos\phi'\,\hat{y} = -\frac{y'}{R}\hat{x} + \frac{x'}{R}\hat{y}$$

$$= -\mathrm{Im}\left(\frac{(x'+iy')}{R}\,\hat{e}_-\right) \quad \text{where } \hat{e}_- = \hat{x}-i\hat{y}$$

$$\Rightarrow \sin\theta'\hat{\phi} = -\frac{N}{R}\mathrm{Im}\left(Y_{1,1}(\theta',\phi')\,\hat{e}_-\right)$$

where $\frac{N}{R}$ is some normalization constant

$$\therefore \int \frac{\sin\theta'\hat{\phi}}{|\vec{x}-\vec{x}'|} d\Omega' = -\frac{N}{R}\mathrm{Im}\left[\int \frac{Y_{1,1}(\theta',\phi')}{|\vec{x}-\vec{x}'|} d\Omega'\, \hat{e}_-\right]$$

Using the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{l,m}(\theta',\phi')\, Y_{l,m}(\theta,\phi)$

and orthonormality $\int Y^*_{l',m'}(\theta',\phi')\, Y_{l,m}(\theta,\phi)\, d\Omega' = \delta_{l,l'}\,\delta_{m,m'}$

only $l=1, m=1$ term

$$\Rightarrow \int \frac{\sin\theta'\hat{\phi}}{|\vec{x}-\vec{x}'|} = \frac{4\pi}{3}\begin{cases} \frac{R}{r^2}\,\mathrm{Im}\left[-\frac{N}{R}Y_{1,1}(\theta,\phi)\,\hat{e}_-\right] & r>R \\ \frac{r}{R^2}\,\mathrm{Im}\left[-\frac{N}{R}Y_{1,1}(\theta,\phi)\,\hat{e}_-\right] & r<R \end{cases}$$

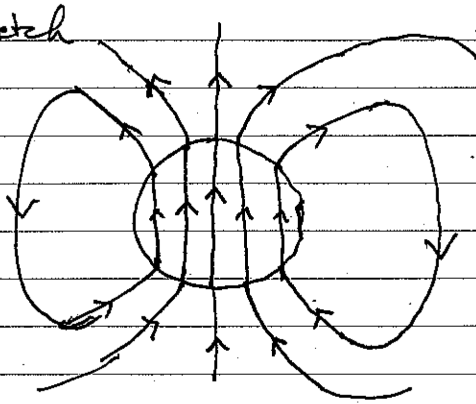
$$= \frac{4\pi}{3}\sin\theta\,\hat{\phi}\begin{cases} \frac{R}{r^2} & r>R \\ \frac{r}{R^2} & r<R \end{cases}$$

$$\boxed{\vec{A}(\vec{x}) = \begin{cases} \frac{4}{3}\pi R^3\left(\frac{\sigma\omega R}{c}\right)\frac{\sin\theta}{r^2}\hat{\phi} & r > R \\ \frac{4\pi}{3}\left(\frac{\sigma\omega R}{c}\right) r\sin\theta\,\hat{\phi} & r < R \end{cases}}$$

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A} = \hat{r}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) + \hat{\theta}\left(-\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\right)$$

$$\Rightarrow \boxed{\vec{B}(\vec{x}) = \begin{cases} \frac{4}{3}\pi R^3\left(\frac{\sigma\omega R}{c}\right)\left[\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right] & r > R \\ \frac{8\pi}{3}\left(\frac{\sigma\omega R}{c}\right)\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) = \frac{8\pi}{3}\left(\frac{\sigma\omega R}{c}\right)\hat{z} & r < R \end{cases}}$$

Sketch



$\vec{B}$: Uniform inside sphere

Pure dipole outside

This can be seen by noting that for $r > R$

$$\vec{A} = \frac{\hat{r}\times\vec{m}}{r^2}$$

where $\vec{m} = \frac{4}{3}\pi R^3\left(\frac{\sigma\omega R}{c}\right)\hat{z}$

Note discontinuity in tangential $\vec{B}$, (i.e. B_θ) of

$\frac{4\pi}{c}\sigma\omega R$

Problem 4 Magnetic Dipole Moment

(a) As in problem 3, we can break up the sphere into rings. The differential moment of each ring is then

$$d\vec{m} = \frac{dI(\theta)}{c}\,(\text{Area of ring})\,\hat{z} = \frac{dI(\theta)}{c}\,\pi(R\sin\theta)^2\hat{z}$$

$$= \frac{\pi\sigma\omega R^4}{c}\sin^3\theta\,d\theta\,\hat{z}$$

The total magnetic moment is

$$\vec{m} = \int d\vec{m} = \pi\frac{\sigma\omega}{c}R^4\int_0^\pi \sin^3\theta\,d\theta\;\hat{z}$$

$$= \pi\frac{\sigma\omega}{c}R^4\int_0^\pi d\theta\,(1-\cos^2\theta)\sin\theta\,d\theta = \pi\frac{\sigma\omega}{c}R^4\underbrace{\int_{-1}^{1}du\,(1-u^2)}_{=4/3}$$

$$\Rightarrow \boxed{\vec{m} = \frac{4\pi R^3}{3}\left(\frac{\sigma\omega R}{c}\right)\hat{z}} \quad \text{As before}$$

(b) Let the radius of the sphere shrink to zero, but the magnetic moment remain constant (σ grows as $\frac{1}{R^4}$)

"Outside" the point the $\vec{B}$ field is pure dipole

$$\vec{B}(\vec{x}) = \frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3} = \frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

What about the field that was inside the sphere

$$r<R \quad \vec{B} = \frac{8\pi}{3}\left(\frac{\sigma\omega R}{c}\right)\hat{z} = \frac{2\vec{m}}{R^3} = \frac{8\pi}{3}\,\frac{\vec{m}}{\left(\frac{4\pi R^3}{3}\right)}$$

In the limit $R\to 0$ $\quad \frac{1}{\frac{4\pi}{3}R^3} \to \delta^{(3)}(\vec{x})$ (Density of a point)

$$\underset{R\to 0}{\lim} \Rightarrow \boxed{\vec{B}(\vec{x}) = \frac{3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}}{r^3} + \frac{8\pi}{3}\vec{m}\,\delta^{(3)}(\vec{x})}$$

This last argument was a little "hand-wavy" but correct. See also Jackson (5.60) – (5.64)

(c) With the mass M ~~contr~~ concentrated on the surface, the moment of inertia is

$$I = \frac{2}{3} M R^2$$

$\Rightarrow$ The classical, orbital angular moment of the spinning sphere is

$$L = I \omega R = \frac{2}{3} M \omega R^2$$

The gyromagnetic ratio

$$\boxed{\gamma = \frac{m}{L} = 2\pi R \left(\frac{\sigma}{M c} R\right) = \frac{Q}{2Mc} \quad \text{where } Q = 4\pi R^2 \sigma \text{ is the total charge}}$$

Note: If $Q = e$ (the charge of the electron)

then $\gamma = \frac{\mu_B}{\hbar}$, where $\mu_B = \frac{e\hbar}{2m_e c}$

Thus a classical model of a spinning electron would g the magnetic moment

$\vec{m} = \mu_B \left(\frac{\vec{L}}{\hbar}\right)$. However $\frac{L}{\hbar} = \frac{1}{2}$ for the electron (spin) while $|\vec{m}| = \mu_B$. Thus, this classical formulae must be replaced by

$\vec{m} = g \mu_B \left(\frac{\vec{S}}{\hbar}\right)$. The "g-factor" is a quantum relativistic effect ≈ 2