Physics 511: Problem Set #6 Solutions

(1) Standing E&M wave

$\vec{k}_1 = k\hat{z}$ $\qquad\qquad$ $\vec{k}_2 = -k\hat{z}$

$\vec{E}_1 = \hat{x}\, E_0 \cos(kz - \omega t)$ $\qquad\qquad$ $\vec{E}_2 = \hat{x}\, E_0 \cos(kz + \omega t)$

(a) $\vec{E}_3 = \vec{E}_1 + \vec{E}_2 = \hat{x}\, E_0 \left(\cos(kz - \omega t) + \cos(kz + \omega t)\right)$

$$= \hat{x}\, E_0 \big(\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t) + \cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)\big)$$

$$\Rightarrow \boxed{\vec{E}_3 = \hat{x}\; 2E_0 \cos(kz)\cos(\omega t)}$$

Alternative solution using complex representation

$\vec{E}_3 = Re(\vec{\underset{\sim}{E}}_3\, e^{-i\omega t})$, $\quad \vec{\underset{\sim}{E}}_3 = \underset{\sim}{E}_1 + \underset{\sim}{E}_2$

$\vec{E}_1 = Re(\vec{\underset{\sim}{E}}_1 e^{-i\omega t}) = \hat{x} E_0 \cos(kz - \omega t) \Rightarrow \vec{\underset{\sim}{E}}_1 = \hat{x} E_0 e^{ikz}$

$\vec{E}_2 = Re(\vec{\underset{\sim}{E}}_2 e^{-i\omega t}) = \hat{x} E_0 \cos(-kz - \omega t) \Rightarrow \vec{\underset{\sim}{E}}_2 = \hat{x} E_0 e^{-ikz}$

$\Rightarrow \vec{\underset{\sim}{E}}_3 = \hat{x} E_0 (e^{ikz} + e^{-ikz}) = \hat{x}\, 2E_0 \sin kz$

$\Rightarrow \vec{E}_3 = Re(\vec{\underset{\sim}{E}}_3\, e^{-i\omega t}) = \hat{x}\, 2E_0 \sin kz\, Re(e^{-i\omega t}) =$
$= \hat{x}\, 2E_0 \sin kz \cos\omega t$ ✓

(b) Since the <u>total field</u> is not a travelling wave, we cannot use $\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E}$

However $\vec{B}_3$ is monochromatic

$$\Rightarrow \vec{B}_3 = Re(\vec{\underset{\sim}{B}}_3\, e^{-i\omega t})$$

Plug this into $\vec{\nabla} \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \frac{i\omega}{c} \underset{\sim}{\vec{B}}_3(z) = \vec{\nabla} \times \underset{\sim}{\vec{E}}_3(z) = \hat{y}\frac{\partial}{\partial z}(2E_0 \cos kz)$$

$$= 2kE_0 \sin kz \; \hat{y}$$

$$\therefore \underset{\sim}{\vec{B}}_3(z) = +2i\frac{kc}{\omega}E_0 \sin kz \; \hat{y} = +2i\, E_0 \sin kz \; \hat{y}$$

$$\therefore \vec{B}_3(z,t) = Re\left(\underset{\sim}{\vec{B}}_3(z)\, e^{-i\omega t}\right) = 2E_0 \sin kz \; Re\left(i e^{-i\omega t}\right)\hat{y}$$

$$\Rightarrow \vec{B}_3(z,t) = 2E_0 \sin kz \sin\omega t \; \hat{y}$$

Alternative solution

Find $\vec{B}_1$ and $\vec{B}_2$ (the mag-field of the two plane waves)

$\vec{B}_1 = \hat{y}\, E_0 \cos(kz - \omega t)$ since $\hat{k}_1 = \hat{z}$

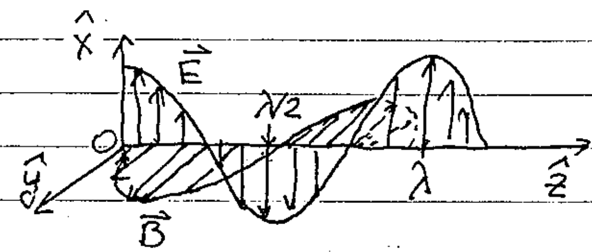
$\vec{B}_2 = -\hat{y}\, E_0 \cos(kz + \omega t)$ since $\hat{k}_2 = -\hat{z}$

$$\vec{B}_3(z,t) = \vec{B}_1 + \vec{B}_2 = \hat{y}\, E_0 \, Re\left(e^{i(kz-\omega t)} + e^{-i(kz+\omega t)}\right)$$

$$= \hat{y}\, E_0 \, Re\left[\left(e^{ikz} - e^{-ikz}\right)e^{-i\omega t}\right] = \hat{y}\, E_0 \, Re\left[2i \sin kz\, e^{-i\omega t}\right]$$

$$= \hat{y}\, E_0 \, 2\sin kz \sin\omega t$$

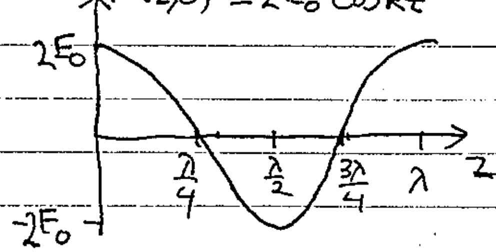
Thus, for a standing wave $\vec{E}$ and $\vec{B}$ are 90° out of phase, both in time and space



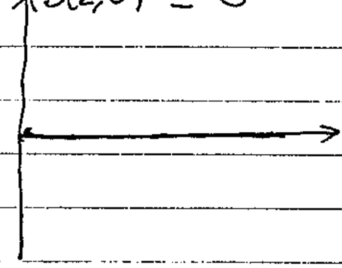
$\vec{E}$ and $\vec{B}$ at $t = T/8$

$t = 0$ and $t = T$

$E(z,0) = 2E_0 \cos kz$

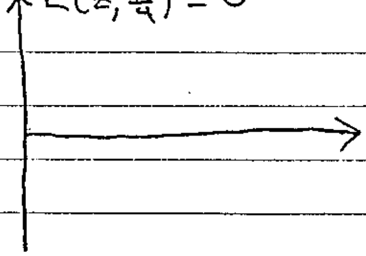


$B(z,0) = 0$

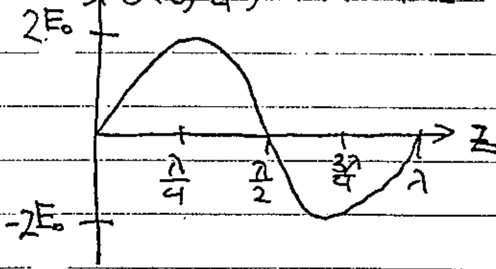


$t = \frac{T}{4}$

$E(z, \frac{T}{4}) = 0$

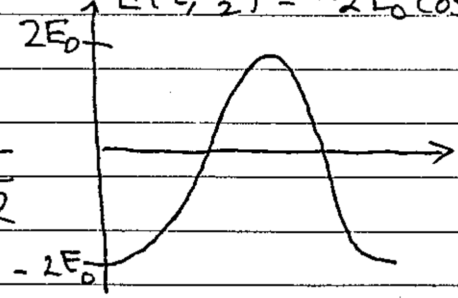


$B(z, \frac{T}{4}) = 2 \sin kz \; E_0$

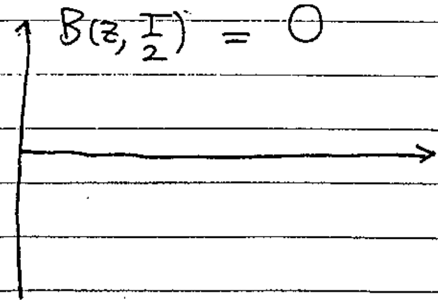


$t = \frac{T}{2}$

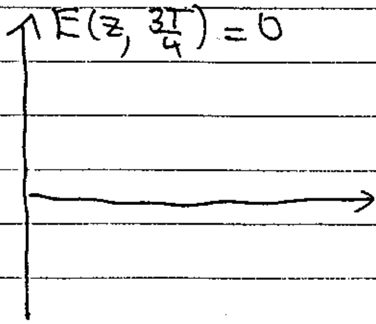
$E(z, \frac{T}{2}) = -2E_0 \cos kz$



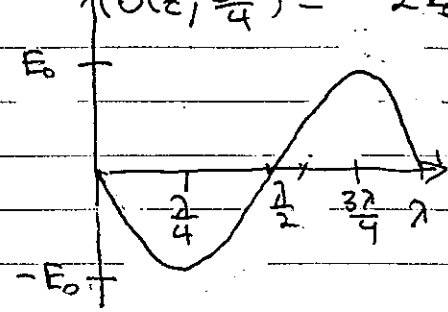
$B(z, \frac{T}{2}) = 0$



$E(z, \frac{3T}{4}) = 0$



$B(z, \frac{3T}{4}) = -2E_0 \sin kz$



Anti-nodes of E at $\frac{m\lambda}{2}$

nodes of E at $\frac{\lambda}{2}(m+\frac{1}{2})$

$m = 0, \pm 1, \ldots$

Anti-nodes of B at $(m+\frac{1}{2})$

nodes of B at $m\frac{\lambda}{2}$

(b) Time averaged electric and magnetic field energy densities

Using real fields:

$$\left\langle u_E(\vec{x}) \right\rangle = \left\langle \frac{\vec{E}^2}{8\pi} \right\rangle = \frac{1}{8\pi} 4\vec{E}_0^2 \cos^2 kz \underbrace{\left\langle \cos^2 \omega t \right\rangle}_{1/2} = \frac{E_0^2}{4\pi} \cos^2 kz$$

$$\left\langle u_B(\vec{x}) \right\rangle = \left\langle \frac{\vec{B}^2}{8\pi} \right\rangle = \frac{1}{8\pi} 4\vec{B}_0^2 \sin^2 kz \left\langle \sin^2 \omega t \right\rangle = \frac{E_0^2}{4\pi} \sin^2 kz$$

Using complex amplitudes $\underset{\sim}{\vec{E}} = (2E_0 \cos kz)\hat{x}$, $\underset{\sim}{\vec{B}} = (i 2E_0 \sin kz)$

$$\left\langle u_E(\vec{x}) \right\rangle = \frac{1}{16\pi} \underset{\sim}{\vec{E}}^* \cdot \underset{\sim}{\vec{E}} = \frac{1}{4\pi} E_0^2 \cos^2 kz$$

$$\left\langle u_B(\vec{x}) \right\rangle = \frac{1}{16\pi} \underset{\sim}{\vec{B}}^* \cdot \underset{\sim}{\vec{B}} = \frac{1}{4\pi} B_0^2 \sin^2 kz$$

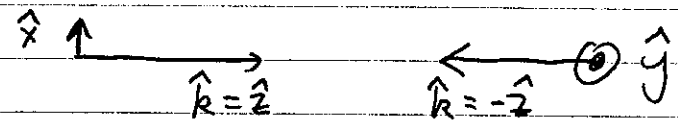
Intensity $\left\langle \vec{S} \right\rangle = \frac{c}{4\pi} \left\langle \vec{E} \times \vec{B} \right\rangle = \frac{c}{4\pi} 4E_0^2 \sin kz \cos kz \left\langle \cancel{\cos\omega t \sin\omega t} \right\rangle$ 0

(Complex amplitude $\left\langle \vec{S} \right\rangle = \mathrm{Re}\left(\frac{c}{8\pi} \underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{B}}^* \right) = \frac{c}{4\pi} 4E_0^2 \, \mathrm{Re}\left(i \sin kz \cos k\right.$

$$\Rightarrow \boxed{\left\langle \vec{S} \right\rangle = 0}$$

$\Rightarrow$ In a standing wave there is no flux of energy as in a traveling wave. Instead, the energy is oscillating back and forth between electric and magnetic fields as in as LC oscillator.

(c) The "lin ⊥ lin" field



$$\vec{E}_1 = \hat{x}\, E_0 \cos(kz-\omega t)$$
$$\vec{E}_2 = \hat{y}\, E_0 \sin(kz-\omega t)$$

Cross polarized counter-propagating plane waves

Complex amplitude representation

- $\vec{E}_1(z,t) = Re\left(\underset{\sim}{\vec{E}}_1(z)\, e^{-i\omega t}\right) = Re\left(\hat{x}\, E_0\, e^{ikz}\, e^{-i\omega t}\right)$

$$\Rightarrow \quad \underset{\sim}{\vec{E}}_1(z) = \hat{x}\, E_0\, e^{ikz}$$

- $\vec{E}_2(z,t) = Re\left(\underset{\sim}{\vec{E}}_2(z)\, e^{-i\omega t}\right) = Re\left(\hat{z}\, i E_0\, e^{-ikz}\, e^{-i\omega t}\right)$

$$\Rightarrow \quad \underset{\sim}{\vec{E}}_2(z) = i\, \hat{z}\, E_0\, e^{-ikz}$$

$\Rightarrow$ Total field $\vec{E}_3(z,t) = Re\left(\underset{\sim}{\vec{E}}_3(z)\, e^{-i\omega t}\right)$

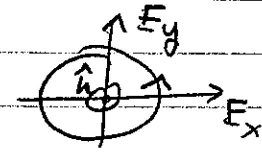
$$\Rightarrow \quad \underset{\sim}{\vec{E}}_3 = \left(e^{ikz}\, \hat{x} + i\, e^{-ikz}\, \hat{y}\right) E_0$$
$$= \left(\frac{\hat{x} + i\, e^{-2ikz}\, \hat{y}}{\sqrt{2}}\right) \sqrt{2}\, E_0\, e^{ikz}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \underset{\sim}{\vec{E}}_3 &= \vec{\epsilon}(z)\, \sqrt{2}\, E_0\, e^{ik_0 z} \\ \vec{\epsilon}(z) &= \frac{\hat{x} + i\, e^{-2ikz}\, \hat{y}}{\sqrt{2}} \end{aligned}}$$

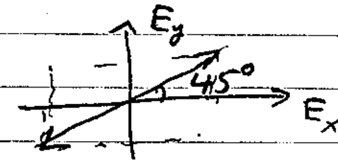
I have written this in such a way that we see the normalized position dependent complex polarization vector $\vec{\epsilon}(z)$

As a function of position

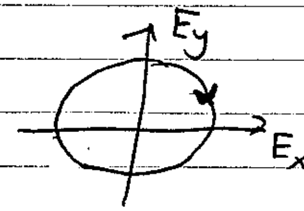
- $\vec{E}(0) = \dfrac{\hat{x}+i\hat{y}}{\sqrt{2}} = \hat{e}_+$: Positive helicity



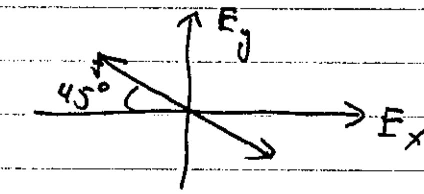
- $\vec{E}\left(\frac{\lambda}{8}\right) = \dfrac{\hat{x}+i\,e^{-i\pi/2}\,\hat{y}}{\sqrt{2}} = \dfrac{\hat{x}+\hat{y}}{\sqrt{2}}$: Linear



- $\vec{E}\left(\frac{\lambda}{4}\right) = \dfrac{\hat{x}+i\,e^{-i\pi}\,\hat{y}}{\sqrt{2}} = \dfrac{\hat{x}-i\hat{y}}{\sqrt{2}}$: Negative Helicity



- $\vec{E}\left(\frac{3\lambda}{8}\right) = \dfrac{\hat{x}+i\,e^{-i\frac{3\pi}{2}}\,\hat{y}}{\sqrt{2}} = \dfrac{\hat{x}-\hat{y}}{\sqrt{2}}$: Linear



- $\vec{E}\left(\frac{\lambda}{2}\right) = \dfrac{\hat{x}+i\,e^{-i2\pi}\,\hat{y}}{\sqrt{2}} = \dfrac{\hat{x}+i\hat{y}}{2}$. Positive Helicity

$\vdots$

etc.

(d) Let us reexpress $\hat{x}$ and $\hat{y}$ in terms of $\hat{e}_\pm = \dfrac{\hat{x}\pm i\hat{y}}{\sqrt{2}}$

$$\Rightarrow \hat{x} = \frac{\hat{e}_+ + \hat{e}_-}{\sqrt{2}}, \qquad \hat{y} = \frac{\hat{e}_+ - \hat{e}_-}{i\sqrt{2}}$$

$$\Rightarrow \vec{\underset{\sim}{E}}_3 = \left(e^{ikz}\left(\frac{\hat{e}_+ + \hat{e}_-}{\sqrt{2}}\right) + i e^{-ikz}\left(\frac{\hat{e}_+ - \hat{e}_-}{i\sqrt{2}}\right)\right) E_0$$

$$= \left[\hat{e}_+\left(\frac{e^{ikz}+e^{-ikz}}{\sqrt{2}}\right) + \hat{e}_-\left(\frac{e^{ikz}-e^{-ikz}}{\sqrt{2}}\right)\right] E_0$$

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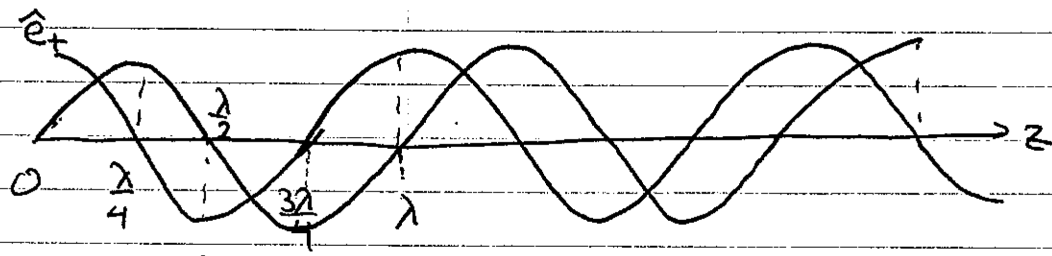
Thus $\vec{\underset{\sim}{E}}_3 = \sqrt{2}\, E_0 \left(\hat{e}_+ \cos kz + i\, \hat{e}_- \,\text{sen}\, kz \right)$

$$\vec{E}_3(z,t) = Re\left(\vec{\underset{\sim}{E}}_3\, e^{-i\omega t}\right) =$$

$$\boxed{\vec{E}_3(z,t) = \hat{e}_+ \sqrt{2} E_0 \cos kz \cos\omega t + \hat{e}_- \sqrt{2} E_0 \sin kz \sin\omega t}$$

Superposition of standing waves of $\hat{e}_\pm$



Node of one standing wave corresponds to anti-nodes of other
(places where $\hat{e}_+$ wave $= \hat{e}_-$ wave $\Rightarrow$ linear polarization)

(c) Intensity as a function of position

$$I = \langle \vec{S} \rangle \quad , \quad \vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B})$$

$$\vec{\underset{\sim}{B}}_3 = \frac{c\vec{k}_1}{\omega} \times \vec{\underset{\sim}{E}}_1 + \frac{c\vec{k}_2}{\omega} \times \vec{\underset{\sim}{E}}_2 = \hat{z} \times \left(\vec{\underset{\sim}{E}}_1 - \vec{\underset{\sim}{E}}_2 \right)$$

$$\Rightarrow \quad \vec{\underset{\sim}{B}}_3(z) = \hat{z} \times \left(e^{ikz} \hat{x} - i e^{-ikz} \hat{y} \right) E_0 = \left(e^{ikz} \hat{y} + i e^{-ikz} \hat{x} \right) E_0$$

$$\Rightarrow I = \frac{c}{8\pi} Re\left(\underset{\sim}{\vec{E}}_3 \times \underset{\sim}{\vec{B}}_3^{*} \right) = \frac{c E_0^2}{8\pi} Re\left[\left(\hat{x} + i e^{-2ikz} \hat{y} \right) \times \left(\hat{y} - i e^{2ikz} \hat{x} \right) \right]$$

$$= \frac{c E_0^2}{8\pi} Re\left[\hat{z} - \hat{z} \right] = \boxed{0} \quad \text{As expected}$$

Note: Since $\vec{E}_1 \perp \vec{E}_2$ $\quad \vec{S} = \vec{S}_1 + \vec{S}_2 = 0$: Orthogonal waves do not interfere

Problem 2: Spherical waves: (Wave fronts on Spheres)

Wave equation for a scalar field $\psi(\vec{r},t)$

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

(a) Monochromatic field whose amplitude depends only on the radial distance $r = |\vec{r}|$

$$\psi(\vec{r},t) = \tilde{\psi}(r)\, e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \tilde{\psi}(r)\, e^{-i\omega t}$$

$$\nabla^2 \psi = e^{-i\omega t}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 \tilde{\psi}(r)) + \underbrace{\partial_\theta^2(\cdots)}_{0} + \underbrace{\partial_\phi^2(\cdots)}_{0}\right)$$

No θ or ϕ dependance

$$\Rightarrow \frac{e^{-i\omega t}}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + \frac{\omega^2}{v^2}\tilde{\psi}(r)\, e^{-i\omega t} = 0$$

$$\rightarrow \boxed{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + k^2 \tilde{\psi}(r) = 0 \quad \text{where } k = \frac{\omega}{v}}$$

Let $\tilde{\psi}(r) = \frac{u(r)}{r}$ (u is known as the "reduced radial" wave function)

$$\frac{\partial \tilde{\psi}}{\partial r} = \frac{u'}{r} - \frac{u}{r^2} \quad \text{where} \quad u' = \frac{du}{dr}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\psi}}{\partial r}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r u' - u\right]$$

$$= \frac{1}{r^2}\left[r^2 u'' + u' - u'\right] = \frac{u''}{r} = \frac{1}{r}\frac{d^2 u}{dr^2}$$

$$= -k^2 \tilde{\psi} = -k^2 \frac{u(r)}{r}$$

$$\Rightarrow \boxed{\frac{d^2 u}{dr^2} + k^2 u = 0}$$

This is nothing other than the simple harmonic oscillator equation.

General solution $\quad u(r) = u_0 e^{\pm ikr} e^{i\phi}$

u_0 and ϕ arbitrary

$$\Rightarrow \psi(r,t) = \mathrm{Re}\left(\frac{u_0}{r} e^{\pm i(kr \pm \omega t + \phi)}\right)$$

$$\Rightarrow \boxed{\psi = u_0 \frac{\cos(kr \pm \omega t + \phi)}{r}}$$

These are spherical waves (ψ constant on a sphere)

$\psi_1 \Rightarrow$ outward propagation, $\psi_2 \Rightarrow$ inward propaga

(b) Vector spherical wave:

Ansatz: $\vec{E} = E_0 \frac{\cos(kr-\omega t)}{kr} \hat{\phi} = E_\phi(r)\,\hat{\phi}$

Transverse? $\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0$ ✓

$\Rightarrow$ $\vec{E}$ must satisfy the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

$$\Rightarrow \quad \nabla^2\vec{E} = -\frac{\omega^2}{c^2}\vec{E} \quad ?$$

$$\nabla^2\vec{E} = \hat{x}(\nabla^2 E_x) + \hat{y}(\nabla^2 E_y)$$ (since $\hat{x}$, $\hat{y}$ independent of position

$$\hat{\phi} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$E_x = \cos\phi\, E_\phi(r) \qquad E_y = \sin\phi\, E_\phi(r)$$

$$\Rightarrow \nabla^2 E_x = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r E_x(r,\phi)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_x(r,\phi)}{\partial\phi^2}$$

$$= -k^2 E_x(r,\phi) - \frac{1}{r^2\sin^2\theta} E_x(r,\phi)$$

$$\Rightarrow \nabla^2\vec{E} = \left(-k^2 - \frac{1}{r^2\sin^2\theta}\right)\vec{E} \neq -\frac{\omega^2}{c^2}\vec{E}$$

$\Rightarrow$ $\vec{E}$ is not a solution to Maxwell's Eqns in free space

(c) Given $\vec{E} = E_0\left(\frac{\sin\theta}{kr}\right)\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{\phi}$

$$= \mathrm{Re}\left(\vec{E}(r,\theta)\, e^{-i\omega t}\right)$$

where $\vec{E}(r,\theta) = E_0 \sin\theta \left(\frac{1}{kr} + \frac{i}{(kr)^2}\right) e^{ikr}\hat{\phi}$ is the complex amplitude

Given the harmonic time dependence $e^{-i\omega t}$, Maxwell's Equations for the complex amplitudes are (in vacuum

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = ik\vec{B}, \quad \vec{\nabla}\times\vec{B} = -ik\vec{E}$$

where $k = \frac{\omega}{c}$

We can find the magnetic field associated with this wave through Faraday's Law

$$\vec{B} = \frac{-i}{k}\vec{\nabla}\times\vec{E} = -\frac{i}{k}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\phi)\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\hat{\theta}\right.$$

Aside: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\phi) = \frac{\partial E_\phi}{\partial\theta} + \frac{\cos\theta}{\sin\theta}E_\phi = 2E_0\cos\theta\left(\frac{1}{kr} + \frac{i}{(kr)^2}\right)$

$$\frac{\partial}{\partial r}(rE_\phi) = \frac{\partial}{\partial r}\left\{\sin\theta\left(\frac{1}{k} + \frac{i}{k^2 r}\right)e^{ikr}\right\} = \left[i - \frac{1}{kr} - \frac{i}{k^2r^2}\right]\sin\theta\, e$$

$$\Rightarrow \vec{B} = \left\{2\cos\theta\left(\frac{1}{(kr)^2} - \frac{i}{kr}\right)\hat{r} + \sin\theta\left(-1 - \frac{i}{kr} + \frac{1}{(kr)^2}\right)\hat{\theta}\right\}\frac{E_0 e^{ik}}{kr}$$

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Thus, the real magnetic field is

$$\boxed{\vec{B} = \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)\left(E_0\frac{\cos(kr-\omega t)}{(kr)^3} + E_0\frac{\sin(kr-\omega t)}{(kr)^2}\right) - E_0\sin\theta\,\frac{\cos(kr-\omega t)}{kr}\,\hat{\theta}}$$

What about the other Maxwell eqns!

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0 \quad (\text{no } \phi \text{ dependence})$$

$$\vec{\nabla}\cdot\vec{B} = -\frac{i}{k}\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E}\right) = 0 \qquad \left[\vec{\nabla}\cdot(\vec{\nabla}\times\ ) = 0\right]$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi}$$

Aside:

$$\frac{\partial}{\partial r}(rB_\theta) = B_\theta + r\frac{\partial B_\theta}{\partial r} =$$

$$= E_0\sin\theta\left[\left(-\frac{1}{kr} - \frac{i}{(kr)^2} + \frac{1}{(kr)^3}\right)e^{ikr} + \left(-i + \frac{1}{kr} + \frac{i}{(kr)^2}\right)e^{ikr} + \left(\frac{1}{kr} + \frac{2i}{k^2r^2} - \frac{3}{(kr)^3}\right)e^{ikr}\right]$$

$$\Rightarrow \frac{\partial}{\partial r}(rB_\theta) = \left(-i + \frac{1}{kr} + \frac{2i}{(kr)^2} - \frac{2}{(kr)^3}\right)E_0\sin\theta\, e^{ikr}$$

$$\frac{\partial B_r}{\partial\theta} = \left(\frac{-2}{(kr)^3} + \frac{2i}{(kr)^2}\right)E_0\sin\theta\, e^{ikr}$$

$$\therefore \vec{\nabla}\times\vec{B} = E_0\sin\theta\left(-\frac{i}{r} + \frac{1}{kr^2}\right)e^{ikr}\,\phi = -ik\vec{E}$$

(d) In the lim $kr << 1$, $\frac{1}{r^3}$ dominates

$$\vec{B} \to \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \frac{E_0 \cos(\omega t)}{(kr)^3}$$

We recognize this as a dipole field, for a dipole along the z-axis

$$3(\hat{z}\cdot\hat{r})\,\hat{r} - \hat{z} = 3\cos\theta\,\hat{r} - \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$
$$= 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$$

$$\Rightarrow \boxed{\vec{B} \to \left(3(\hat{z}\cdot\hat{r})\hat{r} - \hat{z}\right) \frac{E_0 \cos\omega t}{(kr)^3} = \frac{3(\vec{m}(t)\cdot\hat{r})\,\hat{r} - \vec{m}(t)}{r^3}}$$

where $\vec{m}(t) = \frac{E_0}{k^3}\cos\omega t\,\hat{z}$ is the effective magnetic dipole

(e) Time averaged Poynting vector $\langle\vec{S}\rangle = \frac{c}{8\pi}\,\mathrm{Re}(\vec{E}^* \times \vec{B})$

And $\vec{E}^* \times \vec{B} = E_\phi^*\,\hat{\phi} \times (B_r\,\hat{r} + B_\theta\,\hat{\theta}) = E_\phi^* B_r\,\hat{\theta} - E_\phi^* B_\theta\,\hat{r}$

$$E_\phi^* B_r = 2E_0^2 \frac{\sin\theta\cos\theta}{kr}\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)\left(\frac{1}{(kr)^2} - \frac{i}{kr}\right)$$

$$= -2E_0^2 \frac{\sin\theta\cos\theta}{(kr)^3}\, i \left(1 + \frac{1}{(kr)^2}\right) \quad \text{Pure imaginary!}$$

$$E_\phi^* B_\theta = \frac{E_0^2 \sin^2\theta}{kr}\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)\left(-1 - \frac{i}{kr} + \frac{1}{(kr)^2}\right)$$

$$= \frac{E_0^2 \sin^2\theta}{kr}\left(-\frac{1}{kr} - \frac{2i}{(kr)^2} - \frac{i}{(kr)^4}\right)$$

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Thus:

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re}(\vec{E}^* \times \vec{B}) = -\frac{c}{8\pi} \mathrm{Re}(E_\phi^* B_\theta)\,\hat{r}$$

$$\Rightarrow \boxed{\langle \vec{S} \rangle = \frac{c E_0^2 \sin^2\theta}{8\pi (kr)^2}\,\hat{r}}$$

The Poyting vector points in the $\hat{r}$-direction and falls off a $\frac{1}{r^2}$. This is expected for a spherical wave

(~~e~~ f) Energy flux

$$\frac{dW}{dt} = \oint \vec{S} \cdot d\vec{a} = \oint S_r\, r^2 d\Omega$$

where $d\Omega = \cancel{\theta} \sin\theta\, d\theta\, d\phi = -d(\cos\theta)\, d\phi$ is the element of solid angle

Since the field is azimuthally symmetric $d\Omega = -2\pi\, d(\cos\theta$

$$\Rightarrow \frac{dW}{dt} = \int_{-1}^{1} S_r\, 2\pi r^2 d(\cos\theta) = \frac{c}{4} \frac{E_0^2}{k^2} \int_{-1}^{1} (1 - \cos^2\theta)\, d(\cos\theta)$$

$$= \frac{c}{4} \frac{E_0^2}{k^2} \left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_{-1}^{1}$$

$$\Rightarrow \boxed{\frac{dW}{dt} = \frac{c E_0^2}{3k^3}}$$

As excepted the energy flux is independent of the radius of the sphere

Problem 3: Angular Momentum in Electromagnetic ~~Waves~~ Waves

$$\vec{L} = \frac{1}{4\pi c}\int d^3x \; \vec{x} \times (\vec{E}\times\vec{B}) \quad \text{(ang. mom. in field)}$$

(a) $$L_i = \epsilon_{ijk}\,\epsilon_{klm}\,\frac{1}{4\pi c}\int d^3x \; x_j E_l B_m$$

Use $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow B_m = \epsilon_{mnp}\,\partial_n A_p$

$$L_i = \epsilon_{ijk}\,\underbrace{\epsilon_{klm}\,\epsilon_{mnp}}_{\delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln}}\,\frac{1}{4\pi c}\int d^3x \; x_j E_l \partial_n A_p$$

$$\Rightarrow L_i = \epsilon_{ijk}\,\frac{1}{4\pi c}\int d^3x\left[E_l (x_j \partial_k) A_l - x_j E_l \partial_l A_k\right]$$

Aside: $\epsilon_{ijk}\, x_j \partial_k = (\vec{x}\times\vec{\nabla})_i$

Aside: $\int x_j E_l \partial_l A_k \underset{\text{by parts}}{=} -\int \left[\partial_l (x_j E_l)\right] A_k$

$$= -\int \delta_{lj} E_l A_k - \int x_j (\partial_l E_l)\, A_k$$

$$= -\int E_j A_k - \int x_j \underbrace{(\vec{\nabla}\cdot\vec{E})}_{0 \text{ in free space}} A_k$$

$$\Rightarrow L_i = \frac{1}{4\pi c}\int d^3x\left[E_l (\vec{x}\times\vec{\nabla})_i A_l + (\vec{E}\times\vec{A})_i\right]$$

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Thus: $\vec{L} = \vec{L}_{orbital} + \vec{L}_{spin}$

$$\boxed{\begin{aligned} \vec{L}_{orbital} &\equiv \frac{1}{4\pi c}\int d^3x \left(E_\ell \; (\vec{x}\times\vec{\nabla})\, A_\ell \right) \quad \text{(sum over } \ell\text{)} \\ \vec{L}_{spin} &\equiv \frac{1}{4\pi c}\int d^3x \left(\vec{E}(\vec{x}) \times \vec{A}(\vec{x}) \right) \end{aligned}}$$

- The "orbital" angular momentum depends on the spatial variation of the field via $\vec{x}\times\vec{\nabla}$
- The "spin" angular momentum depends on the vector nature of the field ("intrinsic" angular momentum)

(b) Given plane wave decomposition

$$\vec{A}(\vec{x},t) = \sum_{\mu}\int \frac{d^3k}{(2\pi)^3}\left[\tilde{A}_\mu(\vec{k})\, \vec{e}_\mu(\hat{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + c.c. \right]$$

where $\vec{e}_\pm(\hat{k}) = \dfrac{\vec{e}_1 \pm i\,\vec{e}_2}{\sqrt{2}}$ $\qquad \vec{e}_1, \vec{e}_2 \perp$ to $\hat{k}$

$$\vec{E}(\vec{x},t) = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t}$$ (In radiation gauge. The Longitudinal part of $\vec{E}$ does not contribute to $\vec{L}$)

$$\Rightarrow \vec{E}(\vec{x},t) = i\sum_{\mu}\int \frac{d^3k}{(2\pi)^3}\, k\left[\tilde{A}_\mu(\vec{k})\, \vec{e}_\mu(\hat{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} \right] + c.c.$$

$$\vec{A}(\vec{x},t) = \vec{A}^{(+)}(\vec{x},t) + \vec{A}^{(-)}(\vec{x},t)$$

$$\vec{E}(\vec{x},t) = \vec{E}^{(+)}(\vec{x},t) + \vec{E}^{(-)}(\vec{x},t)$$

where $\vec{A}^{(+)} = \sum_{\mu} \int \frac{d^3k}{(2\pi)^3} \tilde{A}_{\mu}(\vec{k}) \vec{e}_{\mu}(\hat{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$= \vec{A}^{(-)*}$$

$$\vec{E}^{(+)} = i \sum_{\mu} \int \frac{d^3k}{(2\pi)} k \tilde{A}_{\mu}(\vec{k}) \vec{e}_{\mu}(\hat{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \vec{E}^{(-)*}$$

$$\Rightarrow \vec{L}_{spin} = \int d^3x \left(\frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} + \frac{\vec{E}^{(+)} \times \vec{A}^{(+)}}{4\pi c} \right) + c.c.$$

Aside:

$$\int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{-i}{4\pi c} \int \frac{d^3k\, d^3k'}{(2\pi)^6} \sum_{\mu,\mu'} k \tilde{A}_{\mu}(\vec{k}) \tilde{A}^*_{\mu'}(\vec{k}') \vec{e}_{\mu}(\hat{k}) \times \vec{e}^*_{\mu'}(\hat{k}')$$

$$\underbrace{\int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} e^{i(\omega_k - \omega_{k'})t}$$

$$= \frac{-i}{4\pi c} \int \frac{d^3k}{(2\pi)^3} k \sum_{\mu} \tilde{A}_{\mu}(\vec{k}) \tilde{A}^*_{\mu'}(\vec{k}) \vec{e}_{\mu}(\hat{k}) \times \vec{e}^*_{\mu'}(\hat{k})$$

Double Aside:

$$\vec{e}_{\mu}(\hat{k}) \times \vec{e}_{\mu'}(\hat{k}) = \pm i \hat{k}\, \delta_{\mu\mu'}$$

$$\vec{e}_{\mu=\pm 1}(\hat{k}) \times \vec{e}_{\mu'=\pm}(\hat{k}) = \pm i \hat{k}$$

$$\Rightarrow \int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{1}{4\pi c} \int \frac{d^3k}{(2\pi)^3} \vec{k} \left[|\tilde{A}_+(\vec{k})|^2 - |\tilde{A}_-(\vec{k})|^2 \right]$$

Note: $\vec{e}_\mu(\hat{k}) \times \vec{e}_\mu(-\hat{k}) = 0$

$$\Rightarrow \int \vec{E}^{(+)} \times \vec{A}^{(+)} \, d^3x = 0$$

$$\therefore \quad \vec{L}_{spin} = \int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} + c.c$$

$$\Rightarrow \boxed{\vec{L}_{spin} = \frac{1}{2\pi c} \int \frac{d^3k}{(2\pi)^3} \, \hat{k} \left[|\tilde{A}_+(\vec{k})|^2 - |\tilde{A}_-(\vec{k})|^2 \right]}$$

Thus, a circularly polarized field with $\vec{e}_\pm = \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ carries angular momentum in the $\pm\hat{k}$ direction (positive/negative helicity)