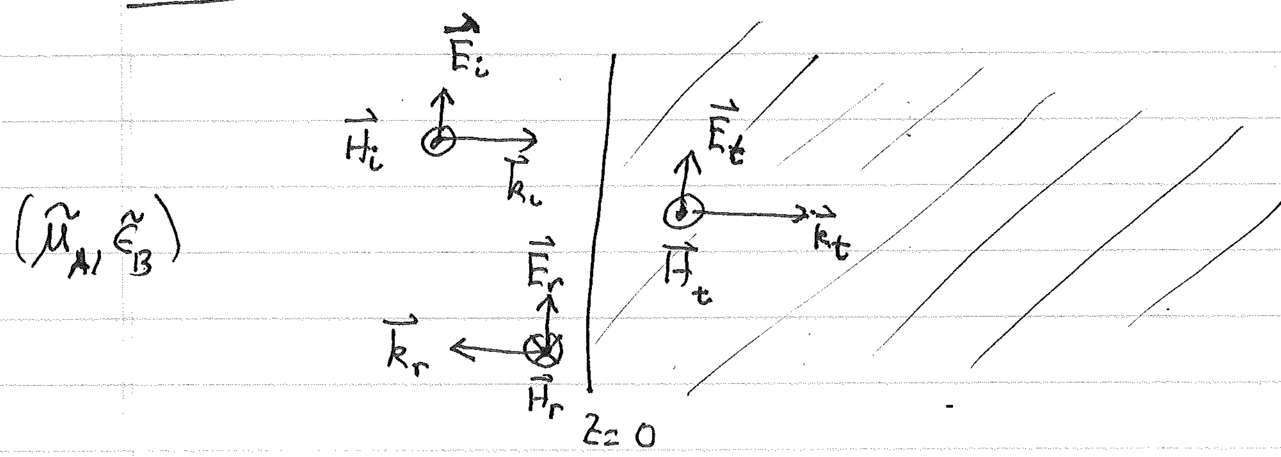
Physics 511: Electrodynamics

Problem Set #8: Solutions

Problem 1: Reflection and transmission



(a) The total wave (complex amplitude)

$$z<0 \quad \begin{cases} \vec{\underset{\sim}{E}}(z) = \vec{\underset{\sim}{E}}_i(z) + \vec{\underset{\sim}{E}}_r(z) = \hat{e}\left(\underset{\sim}{E}_i e^{i\underset{\sim}{k}_i z} + \underset{\sim}{E}_r e^{-i\underset{\sim}{k}_i z}\right) \\ \vec{\underset{\sim}{H}}(z) = \vec{\underset{\sim}{H}}_i(z) + \vec{\underset{\sim}{H}}_r(z) = (\hat{k}\times\hat{e})\left(\underset{\sim}{H}_i e^{i\underset{\sim}{k}_i z} - \underset{\sim}{H}_r e^{-i\underset{\sim}{k}_i z}\right) \end{cases}$$

$$z>0 \quad \vec{\underset{\sim}{E}}(z) = \hat{e}\, \underset{\sim}{E}_t e^{i\underset{\sim}{k}_t z}, \quad \vec{\underset{\sim}{H}}(z) = (\hat{k}\times\hat{e})\, \underset{\sim}{H}_t e^{i\underset{\sim}{k}_t z}$$

Boundary conditions: $\Delta D_\perp = 4\pi\sigma_f \qquad \Delta B_\perp = 0$

$\Delta E_{\parallel} = 0 \qquad \Delta H_{\parallel} = \frac{4\pi}{c} K_f$

In this case the components of $\vec{E}$ & $\vec{H}$ are parallel to the interface @ and there is no free surface charge nor surface current

$$\Delta E_{\parallel} = 0 \Rightarrow \underset{\sim}{E}_i + \underset{\sim}{E}_r = \underset{\sim}{E}_t$$

$$\Delta H_{\parallel} = 0 \Rightarrow \underset{\sim}{H}_i - \underset{\sim}{H}_r = \underset{\sim}{H}_t$$

We define the amplitude reflection/transmission coefficients: $\tilde{r} \equiv \dfrac{\tilde{E}_r}{\tilde{E}_i}$, $\tilde{t} = \dfrac{\tilde{E}_t}{\tilde{E}_i}$

$\Rightarrow \quad 1 + \tilde{r} = \tilde{t}$

Also, for a plane wave $\tilde{H} = \tilde{E} / \tilde{Z}$ where $\tilde{Z} = \sqrt{\dfrac{\tilde{\mu}}{\tilde{\epsilon}}}$

$\Rightarrow \quad \dfrac{1 - \tilde{r}}{\tilde{Z}_A} = \dfrac{\tilde{t}}{\tilde{Z}_B} \Rightarrow \tilde{t} = \dfrac{\tilde{Z}_B}{\tilde{Z}_A}(1 - \tilde{r}) = 1 + \tilde{r}$

Solving for $\tilde{r}$: $\tilde{r}\left(1 + \dfrac{\tilde{Z}_B}{\tilde{Z}_A}\right) = \dfrac{\tilde{Z}_B}{\tilde{Z}_A} - 1$

$$\therefore \quad \boxed{\tilde{r} = \frac{\tilde{Z}_B - \tilde{Z}_A}{\tilde{Z}_A + \tilde{Z}_B}}$$

Plug back into $\tilde{t}$

$$\boxed{\tilde{t} = 1 + \tilde{r} = \frac{2\tilde{Z}_B}{\tilde{Z}_A + \tilde{Z}_B}}$$

Reflection from an interface is due to an impedance mismatch. This is a general wave phenomenon.

(b) for non-magnetic, non-absorbing medium

$$\tilde{Z} = \frac{1}{\sqrt{\epsilon}} \quad \text{(real)} \quad = \frac{1}{n} \leftarrow \text{index of refraction}$$

$$\Rightarrow \boxed{\begin{aligned} r &= \frac{n_B^{-1} - n_A^{-1}}{n_A^{-1} + n_B^{-1}} = \frac{n_A - n_B}{n_A + n_B} \\ t &= \frac{2 n_B^{-1}}{n_A^{-1} + n_B^{-1}} = \frac{2 n_A}{n_A + n_B} \end{aligned}}$$

(c) Define the coefficients for reflection and transmission of intensity: $R \equiv \frac{I_r}{I_i}$, $T \equiv \frac{I_t}{I_i}$

Recall the intensity of a plane wave:

$$I = \left| \frac{c}{8\pi} \mathrm{Re}\left(\tilde{\vec{E}} \times \tilde{\vec{H}}^* \right) \right| = \frac{c}{8\pi} \mathrm{Re}(\tilde{Z}^{-1}) |\tilde{\vec{E}}|^2$$

For non-magnetic material $\hat{\mu} = 1$ $\quad I = \frac{c}{8\pi} \mathrm{Re}(\hat{n}(\omega)) |\tilde{\vec{E}}|^2$

If $\hat{n}(\omega)$ is real $\quad I = \frac{c}{8\pi} n(\omega) |\vec{E}|^2$

Thus $\quad R = \dfrac{\frac{c}{8\pi} n_A |\tilde{\vec{E}}_r|^2}{\frac{c}{8\pi} n_A |\tilde{\vec{E}}_i|} = |\tilde{r}|^2 = \dfrac{n_A^2 + n_B^2 - 2n_A n_B}{(n_A + n_B)^2}$

$$T = \frac{\frac{c}{8\pi} n_B |\tilde{\vec{E}}_t|^2}{\frac{c}{8\pi} n_A |\tilde{\vec{E}}_i|^2} = \frac{n_B}{n_A} |\tilde{t}|^2 = \frac{4 n_A n_B}{(n_A + n_B)^2}$$

thus, $R + T = \dfrac{n_A + n_B^2 + 2 n_A n_B}{(n_A + n_B)^2} = 1$

as expected

(d) In the general case, with $\tilde{n}_A = 1$, $\tilde{n}_B = \tilde{n}(\omega)$

$$R = |\tilde{r}|^2 = \left| \frac{1 - \tilde{n}(\omega)}{1 + \tilde{n}(\omega)} \right|^2$$

$$T = \frac{\mathrm{Re}(\tilde{n}(\omega))}{1} |\tilde{t}|^2 = \frac{\mathrm{Re}(\tilde{n}(\omega))\, 4}{|1 + \tilde{n}(\omega)|^2}$$

(e) According to Poyntings theorem, with no current $\vec{J}$ in the macroscopic description

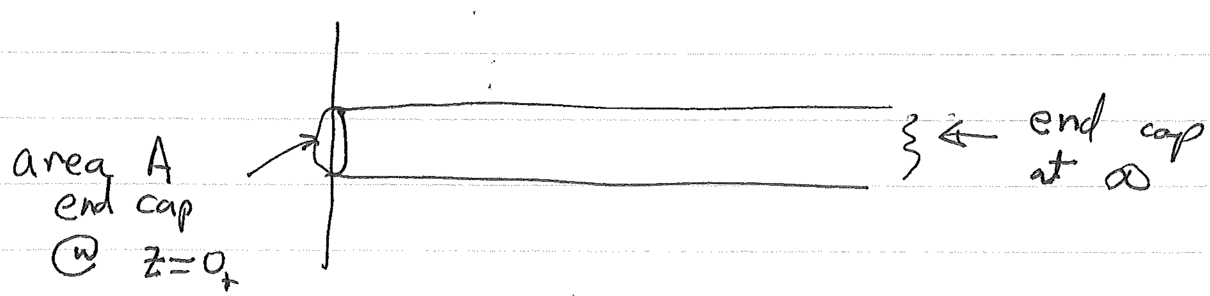
$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} = 0 \quad , \quad \vec{\nabla} \cdot \vec{S} = \frac{c}{4\pi} \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi} \left(\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right)$$

In integral form:

$$\oint_S \vec{S} \cdot \hat{n}\, da = - \int_V d^3x \left(\frac{\partial u}{\partial t} \right)$$

We seek to use this to determine I_T. To do so we choose a surface



$$\oint \langle \vec{S} \rangle \cdot \hat{n}\, da = -A I_t = -\text{Transmitted Power}$$

Note: Because the medium is attenuating, the flux at the endcap at $\infty = 0$.

The loss of power (attenuation) must be seen in the reduction of energy density along the ~~transmiss~~ transmission distance $z = 0 \to \infty$

$$-\int_V d^3x \left\langle \frac{\partial u}{\partial t} \right\rangle = -A \int_0^\infty dz \left\langle \frac{\partial u}{\partial t} \right\rangle$$

$$\left\langle \frac{\partial u}{\partial t} \right\rangle = \frac{1}{8\pi} \mathrm{Re}\left\{ (-i\omega) \left(\vec{\underset{\sim}{E}}^* \cdot \underset{\sim}{\vec{D}} + \vec{\underset{\sim}{H}}^* \cdot \underset{\sim}{\vec{B}} \right) \right\}$$

$$= \frac{1}{8\pi} \mathrm{Re}\left\{ (-i\omega) \left(\tilde{\epsilon}(\omega) \left| \vec{\underset{\sim}{E}}(z) \right|^2 + \underset{\tilde{\mu}=1}{\uparrow} \left| \vec{\underset{\sim}{B}}(z) \right|^2 \right) \right\}$$

$$= \frac{\omega}{8\pi} \mathrm{Im}(\tilde{\epsilon}(\omega)) \left| \vec{\underset{\sim}{E}}_t \right|^2 \underbrace{e^{2\,\mathrm{Im}(\tilde{k}_t) z}}_{\text{attenuation}}$$

$$\left[\begin{array}{l} \text{Note: } \mathrm{Im}(\tilde{k}_t) = \mathrm{Im}(\tilde{n}(\omega)) \frac{\omega}{c} \\ \mathrm{Im}(\tilde{\epsilon}(\omega)) = \mathrm{Im}(\tilde{n}(\omega)^2) = 2\,\mathrm{Re}(\tilde{n}(\omega))\,\mathrm{Im}(\tilde{n}(\omega)) \end{array} \right.$$

$$\Rightarrow -\int_V d^3x \left\langle \frac{\partial u}{\partial t} \right\rangle = -\frac{A\omega}{4\pi} \mathrm{Re}(\tilde{n}(\omega))\,\mathrm{Im}(\tilde{n}(\omega)) \left| \vec{\underset{\sim}{E}}_t \right|^2 \int_0^\infty dz\, e^{-\frac{2\omega z}{c} \mathrm{Im}(\tilde{n})}$$

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$$\Rightarrow -\int_V d^3x \left\langle \frac{\partial u}{\partial t} \right\rangle = -A \frac{c\omega}{4\pi} \,\mathrm{Re}(\tilde{n}(\omega))\, \mathrm{Im}(\tilde{n}(\omega))\, |\tilde{\vec{E}}_t|^2 \left[\frac{e^{-2\,\mathrm{Im}(\tilde{k})z}}{-2\frac{\omega}{c}\,\mathrm{Im}(\tilde{n}(\omega))} \right]_0^\infty$$

$$= -A \frac{c}{8\pi} \mathrm{Re}(\tilde{n}(\omega))\, \underbrace{|\tilde{E}_t|^2}_{|t|^2\,|\tilde{E}_0|^2}$$

$$= -A \frac{4\,\mathrm{Re}(\tilde{n}(\omega))}{|1+\tilde{n}(\omega)|^2} \left(\frac{c}{8\pi} |\tilde{E}_0| \right)^2 = -A\,T\,I_i \checkmark$$

q.e.d.

(f) We consider the case that medium A is the vacuum and medium B is a good conductor. We look @ frequencies $\omega << \sigma_0$ so $|\epsilon(\omega)| >> 1$

$$\tilde{\epsilon}(\omega) \cong i\,\frac{4\pi\sigma_0}{\omega} \Rightarrow \tilde{n}(\omega) = \sqrt{\tilde{\epsilon}(\omega)} = e^{i\frac{\pi}{4}} \sqrt{\frac{4\pi\sigma_0}{\omega}}$$

$$R = \left| \frac{1-\tilde{n}(\omega)}{1+\tilde{n}(\omega)} \right|^2 = \left| \frac{1 - e^{i\frac{\pi}{4}}\sqrt{\frac{4\pi\sigma_0}{\omega}}}{1 + e^{i\frac{\pi}{4}}\sqrt{\frac{4\pi\sigma_0}{\omega}}} \right|^2 = \left| \frac{e^{-i\frac{\pi}{4}}\sqrt{\frac{\omega}{4\pi\sigma_0}} - 1}{e^{-i\frac{\pi}{4}}\sqrt{\frac{\omega}{4\pi\sigma_0}} + 1} \right|^2$$

$$\approx \left| \left(e^{-i\frac{\pi}{4}} \sqrt{\frac{\omega}{4\pi\sigma_0}} - 1 \right) \left(1 - e^{-i\frac{\pi}{4}} \sqrt{\frac{\omega}{4\pi\sigma_0}} \right) \right|^2 \quad \text{for } \frac{\omega}{4\pi\sigma_0} << 1$$

$$\approx \left| 1 - e^{-i\frac{\pi}{4}} \sqrt{\frac{\omega}{\pi\sigma_0}} \right|^2 \approx \left| \left(1 - \sqrt{\frac{\omega}{2\pi\sigma_0}} \right) + i \sqrt{\frac{\omega}{2\pi\sigma_0}} \right|^2$$

$$\Rightarrow \boxed{R \approx 1 - \sqrt{\frac{2\omega}{\pi\sigma_0}}}$$

For silver, $\sigma_0 = 6.14 \times 10^7 \ \text{ohm.m}^{-1}$ (SI units)
$= 5.526 \times 10^{17} \ s^{-1}$ (cgs Gaussian)

THz radiation @ $\lambda = 25 \mu m \Rightarrow \omega = \frac{2\pi}{\lambda} c = 7.4 \times 10^{13} s^{-1}$

$$\Rightarrow \boxed{1 - R = 9.2 \times 10^{-3}}$$

This says that close to 1% of the incident energy is absorbed by the metal.

Note: This calculation holds only for $\omega \ll \sigma_0$. At higher frequencies we must include the dispersion of $\tilde{\sigma}(\omega)$. When $\omega \gg \nu_e$
collision

the metal looks like a plasma. then, if $\omega < \omega_p$ we have essentially 100% reflection and if $\omega > \omega_p$ the metal become quite transparent and behaves like a dielectric.

Problem 2: Group-Velocity dispersion

We consider a quasimonochromatic pulse whose positive frequency component is $\vec{E}^{(+)}(z,t) = \vec{\mathcal{E}}(z,t)\, e^{i(k_0 z - \omega_0 t)}$

where $k_0 = \frac{\omega_0}{c} n(\omega_0)$, $\vec{\mathcal{E}}(z,t)$ is the "slowly-varying pulse envelope" with $\left|\frac{\partial \vec{\mathcal{E}}}{\partial z}\right| \ll k_0 |\vec{\mathcal{E}}|$, $\left|\frac{\partial \vec{\mathcal{E}}}{\partial t}\right| \ll \omega_0 \vec{\mathcal{E}}$

In the frequency domain

$$\vec{\tilde{E}}^{(+)}(z,\omega) = \vec{\tilde{\mathcal{E}}}(z, \omega - \omega_0)\, e^{ik_0 z}$$

(a) In the frequency domain, $\vec{\tilde{E}}^{(+)}$ satisfies the Helmhotz equation (here for one spatial coördinate)

$$\left(\frac{\partial^2}{\partial z^2} + k^2(\omega)\right) \vec{\tilde{E}}^{(+)}(z,\omega) = 0$$

Plugging in the Ansatz in terms of the SVE

$$\Rightarrow \frac{\partial^2}{\partial z^2}\left(\vec{\tilde{\mathcal{E}}}\, e^{ik_0 z}\right) + k^2(\omega)\, \vec{\tilde{\mathcal{E}}} = 0$$

$$\left(k^2(\omega) - k_0^2\right) \vec{\tilde{\mathcal{E}}} + 2ik_0 \frac{\partial \vec{\tilde{\mathcal{E}}}}{\partial z} + \underbrace{\cancel{\frac{\partial^2 \vec{\tilde{\mathcal{E}}}}{\partial z^2}}}_{\approx 0} = 0$$

neglect in SVEA

Now, under the quasimonochromatic assumption we can Taylor-expand $k(\omega)$ around ω_0

$$k(\omega) \approx \underbrace{k(\omega_0)}_{k_0} + (\omega - \omega_0) \left.\frac{dk}{d\omega}\right|_{\omega_0} + \frac{1}{2}(\omega - \omega_0)^2 \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0} + \dots$$

$$\Rightarrow k^2(\omega) \approx \left(k_0 + \frac{\Delta}{v_g} + \frac{1}{2}\Delta^2 k_0''\right)^2$$

where $v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$ $\quad k_0'' = \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}$

$\Delta = \omega - \omega_0$

$$\Rightarrow k^2(\omega) \approx k_0^2 + 2k_0\frac{\Delta}{v_g} + k_0\Delta^2 k_0''$$

Under the assumption the $\underset{\substack{\uparrow \\ \text{large!}}}{k_0} k_0'' \gg \frac{1}{v_g^2}$

$$\Rightarrow \left(2k_0\frac{\Delta}{v_g} + k_0\Delta^2 k_0''\right)\vec{\tilde{\mathcal{E}}} + 2ik_0\frac{\partial\vec{\tilde{\mathcal{E}}}}{\partial z} = 0$$

$$\Rightarrow \boxed{i\frac{\partial\vec{\tilde{\mathcal{E}}}}{\partial z} + \left(\frac{1}{v_g}\Delta + \frac{k_0''}{2}\Delta^2\right)\vec{\tilde{\mathcal{E}}}(z,\Delta) = 0}$$

(b) We take the inverse Fourier transform

$$\vec{\mathcal{E}}(z,t) = \int_{-\infty}^{\infty}\frac{d\Delta}{2\pi}\,\vec{\tilde{\mathcal{E}}}(z,\Delta)\,e^{-i\Delta t}$$

$$\Rightarrow \Delta \Rightarrow i\frac{\partial}{\partial t}$$

$$\therefore\ i\frac{\partial\vec{\mathcal{E}}}{\partial t} + \frac{i}{v_g}\frac{\partial\vec{\mathcal{E}}}{\partial t} - \frac{k_0''}{2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} = 0$$

or $\boxed{\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)\vec{\mathcal{E}} = -i\frac{k_0''}{2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}}$

(c) Now we change variables. Define the "retarded time" w.r.t. v_g : $\tau = t - \frac{z}{v_g}$

Let $A(\tau, z) = \hat{e}\,\mathcal{E}(z, t = \tau + \frac{z}{v_g})$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial A}{\partial z} &= \frac{\partial \mathcal{E}}{\partial z} + \frac{\partial \mathcal{E}}{\partial t}\frac{\partial t}{\partial z} \\ &= \frac{\partial \mathcal{E}}{\partial z} + \frac{1}{v_g}\frac{\partial \mathcal{E}}{\partial t} = -\frac{i k_0''}{2}\frac{\partial^2 A}{\partial \tau^2} \end{aligned}}$$

W.r.t. the new variables, $A(\tau, z)$ satisfies an equation of motion that is isomorphic to the time-dependent Schrödinger equation for a free particle in 1D

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m}\right)\psi = -i\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

Here the independent variable is the distance of propagation z and the "position variable" is τ, the retarded time.

Given an initial Gaussian wave packet in quantum mechanics

$$\psi(x,0) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{x^2}{4\sigma_0^2}}$$

So $P(x,0) = |\psi(x,0)|^2 = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{x^2}{2\sigma_0^2}}$

^ Probability density

Then $\psi(x,t) = \frac{1}{(2\pi)^{1/4}} \sqrt{\frac{\sigma_0}{\sigma_0^2 + i\frac{\hbar t}{2m}}} \, e^{\frac{-x^2}{4(\sigma_0^2 + i\frac{\hbar t}{2m})}}$

so $|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \, e^{\frac{-x^2}{2\sigma^2(t)}}$

Where $\sigma^2(t) = \sigma_0^2 + \frac{\hbar^2}{4\sigma_0^2 m^2} t^2$

The spread of the wave packet width is easily understood. Given an initial spread $\Delta x(0) = \sigma_0^2$, there is a spread in momentum $\Delta p(0) = \frac{\hbar}{2\Delta x(0)}$ (minimum uncertainty)

This implies a variance in arrival of the peak that adds to the initial spread: $\Delta x_{dispersion} = \left(\frac{\Delta p(0) t}{m}\right)^2 = \frac{\hbar^2}{4\sigma_0^2 m} t^2$

This directly translates into the dispersion of an electromagnetic pulse. Given $\mathcal{E}(z=0,t) = \mathcal{E}_0 e^{\frac{-t^2}{2\Delta t_0^2}}$

$\Rightarrow A(\tau, z=0) = \mathcal{E}_0 e^{\frac{-\tau^2}{4\Delta t_0^2}}$

$\Rightarrow A(\tau, z) = \mathcal{E}_0 \sqrt{\frac{\beta(z)}{\Delta t_0^2 + i\beta(z)}} \, e^{\frac{-\tau^2}{4(\Delta t_0^2 + i\beta)}}$

$$\beta = \frac{k_0''}{\Delta t_0} z = \left|\Delta\left(\frac{z}{v_g}\right)\right|$$

$\Rightarrow$ Spread of width $\Delta t^2(z) = \Delta t_0^2 + \frac{(k_0'')^2}{\Delta t_0^2} z^2$

(d) 20 ps pulse at $\lambda = 1.55 \mu m$

Characteristic distance of spreading

$$z_{spread} = \frac{\Delta t(0)^2}{\left|\frac{d^2k}{d\omega^2}\right|} = \frac{400 ps^2}{25\ ps^2/km} = \boxed{16\ km}$$

The power is attenuated by $P(z) = e^{-2\beta z} P(0)$

$\Rightarrow$ The characteristic attenuation length

$$z_{attenuation} = \frac{1}{2\beta} = \frac{1}{2 \times 10^{-5}\ cm^{-1}} = 5 \times 10^4 cm = \boxed{0.5\ km}$$

Yes attenuation is a problem over distances of a few kilometers. This can be overcome through the use of light amplifiers (laser) directly inside the fiber!

Note, however, how incredibly transparent glass is at this frequency in any event

(e) 20 ps pulse at 10 Gbits/sec = Bit rate

Time window $T = \frac{1}{\text{Bit rate}} = \frac{1}{10 \times 10^9\ sec^{-1}} = 10^{-10}$ sec

$= 100$ ps

For a bit-rate-error $< 10^{-9}$, we require pulse duration

$$\Delta t < T/2 = 50\ ps$$

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As a functions of propagation distance, the pulse width increases according to

$$\Delta t(z) = \sqrt{(\Delta t(0))^2 + \left(\frac{k'' z}{\Delta t(0)}\right)^2}$$

$$= \sqrt{1 + \left(\frac{z}{z_{spread}}\right)^2}\ \Delta t(0)$$

Maximum propagation distance: z_{max}

$$\Delta t(z_{max}) = \sqrt{1 + \left(\frac{z_{max}}{z_{spread}}\right)^2}\ \Delta t(0) < \frac{T}{2}$$

$$\Rightarrow z_{max} = \sqrt{\left(\frac{T}{2\Delta t(0)}\right)^2 - 1}\ z_{spread}$$

$$\Rightarrow z_{max} = \sqrt{\left(\frac{50}{25}\right)^2 - 1}\ z_{spread} \approx 4.6\ z_{spread}$$

$$\boxed{z_{max} \approx 73.6 \text{ km}}$$

This is not so bad for some local data transmissions, but it's deadly for long distance communications. One solution to this problem is to make use of some nonlinear effects in the fiber which can counteract the spreading. These stable, non-spreading, pulses are known as "solitons" and appear in a variety of nonlinear dynamical systems. Other solutions include engineering the fiber to reduce dispersion near the low loss window, and cleverer schemes for coding the data. These are hot topics in optical communications

Problem 3: Negative Group Velocity

We consider the dispersion of a wavepacket whose spectral content lines between two gain-lines. In this region we can have high transparency but anomalous dispersion and a negative group velocity!

We begin with the polarizability of a Lorentz oscillator with oscillator strength = -1 (amplifier for inverted atoms)

$$\tilde{\alpha} = \sum_j \frac{e^2}{m} \frac{-1}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

(Resonance freq. ω_j, linewidth γ_j)

$\Rightarrow$ Complex index of refraction (density N, no local-field corrections)

$$\tilde{n}(\omega) = \sqrt{1 + 4\pi N \tilde{\alpha}(\omega)} \approx 1 + 2\pi N \tilde{\alpha}(\omega)$$

$$= 1 - \frac{\omega_p^2/2}{\omega_1^2 - \omega^2 - i\gamma\omega} - \frac{\omega_p^2/2}{\omega_2^2 - \omega^2 - i\gamma\omega}$$

where $\omega_p^2 = \frac{4\pi N e^2}{m}$ is the square of the plasma freq

and we have two resonances @ $\omega_1 = \omega_0 + \Delta/2$
$\omega_2 = \omega_0 - \Delta/2$

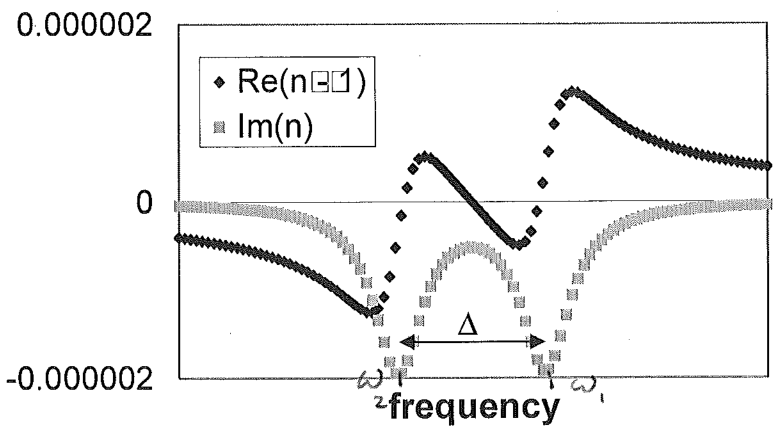
the linewidth of the resonances are taken to be equal $\gamma_1 = \gamma_2 = \gamma$

(a) We consider the parameters

$$\frac{\omega_0}{2\pi} = 352\,\mathrm{THz}, \quad \frac{\Delta}{2\pi} = 1.8\,\mathrm{MHz}, \quad \frac{\gamma}{2\pi} = 220\,\mathrm{kHz}$$

$$\frac{\omega_p}{2\pi} = 21.6\,\mathrm{MHz}$$

A Plot is shown below



Note For $\omega_2 < \omega < \omega_1$ we have anomalous dispersion and essential transparent band.

(b) Let us consider the real part of $\tilde{n}(\omega)$

$$\mathrm{Re}(\tilde{n}(\omega)) = 1 - \frac{\omega_p^2}{2}\left[\frac{\omega_1^2 - \omega^2}{(\omega_1^2-\omega^2)^2 + \gamma^2\omega^2} + \frac{\omega_2^2 - \omega^2}{(\omega_2^2-\omega^2)^2 + \gamma^2\omega^2}\right]$$

$$\approx 1 - \frac{\omega_p^2}{2}\left[\frac{\omega_0^2 + \Delta\omega_0 - \omega^2}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} + \frac{\omega_0^2 - \Delta\omega_0 - \omega^2}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2}\right]$$

Having used $|\Delta| \ll \omega_0$

We seek the "Group Index" $n_g = \frac{d}{d\omega}\left(\omega \, \mathrm{Re}(\hat{n}(\omega))\right)$

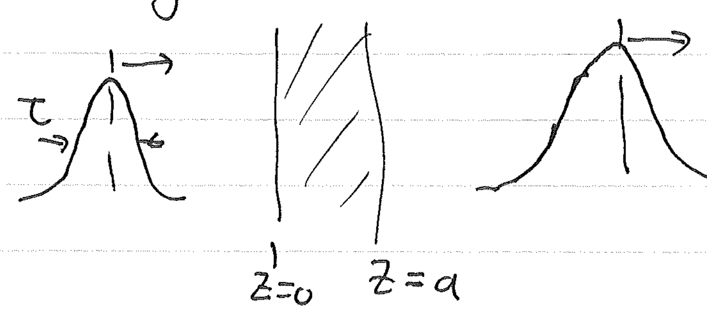
After some tedious algebra, (or using Mathematica), we find

$$\boxed{n_g \simeq 1 - \frac{2\omega_p^2(\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^2}}$$

the group velocity

$$\boxed{v_g = \frac{c}{n_g} = c\left[1 - \frac{2\omega_p^2(\Delta^2-\gamma^2)}{(\Delta^2+\gamma^2)^2}\right]^{-1} \simeq \underbrace{-\frac{c}{2}\frac{\Delta^2}{\omega_p^2}}_{\text{negative!}} \quad \text{for } \gamma \ll \Delta \ll \omega_0}$$

(c) We now consider a Gaussian pulse incident on the gain medium



@ $z=0$ $\quad E^{(+)}(t) = E_0 \, e^{-\frac{t^2}{2\tau^2}} e^{-i\omega_0 t}$ (Gaussian, width τ)

$$\Rightarrow \tilde{E}(\omega) = \underbrace{\tilde{\mathcal{E}}(\omega - \omega_0) + \tilde{\mathcal{E}}(\omega + \omega_0)}_{\text{Gaussian, centered @ } \pm\omega_0}$$

- For $z < 0$ we have free propagation of the wave packet in vacuum: $k = \frac{\omega}{c}$

$$E^{(+)}(z,t) \cong \int_0^\infty \frac{d\omega}{2\pi} \tilde{\mathcal{E}}(\omega-\omega_0) e^{-i\omega t} e^{i\frac{\omega}{c}z}$$

$$\cong \underbrace{\int_{-\infty}^{\infty} \frac{d\Delta}{2\pi} \tilde{\mathcal{E}}(\Delta)\, e^{-i\Delta(t-\frac{z}{c})}}_{E^{(+)}(z=0,\, t-z/c)} \quad e^{-i\omega_0(t-\frac{z}{c})}$$

$$\boxed{z<0 \quad E^{(+)}(z,t) = E_0\, e^{-\frac{(t-z/c)^2}{2\tau^2}} \quad e^{i\omega_0(z/c - t)}}$$

- For $0<z<a$ we have propagation in the medium

$$\tilde{k}(\omega) = \tilde{n}(\omega)\frac{\omega}{c}$$

$$\Rightarrow E^{(+)}(z,t) \cong \int_0^\infty \frac{d\omega}{2\pi} \tilde{\mathcal{E}}(\omega-\omega_0)\, e^{-i\omega t} \quad e^{i\tilde{k}(\omega)z}$$

$$\cong \int_{-\infty}^{\infty} \frac{d\Delta}{2\pi} \tilde{\mathcal{E}}(\Delta)\, e^{-i\Delta t}\, e^{i\tilde{k}(\omega_0+\Delta)z} \quad e^{-i\omega_0 t}$$

Aside $\tilde{k}(\omega_0+\Delta) \approx \tilde{k}(\omega_0) + \Delta \underbrace{\left.\frac{d\tilde{k}}{d\omega}\right|_{\omega_0}}_{=\frac{1}{v_g}}$

$$\Rightarrow E^{(+)}(z,t) \cong \int_{-\infty}^{\infty} \frac{d\Delta}{2\pi} \tilde{\mathcal{E}}(\Delta)\, e^{-i\Delta(t-z/v_g)}\, e^{i(\tilde{k}(\omega_0)z-\omega_0 t)}$$

$$\boxed{\text{Thus, for } 0<z<a \qquad E^{(+)}(z,t) = E_0\, e^{\frac{-(t-z/v_g)^2}{2\tau^2}}\, e^{i\omega_0\left(\frac{\tilde{n}(\omega_0)z}{c} - t\right)}}$$

For $z>a$, we again propagate in vacuum, but all of the Fourier components are phase-shifted by an amount depending on the dispersion relation

$$\Rightarrow \text{For } z>a, \quad E^{(+)}(z,t) \cong \int_0^\infty \frac{d\omega}{2\pi}\, \tilde{\mathcal{E}}(\omega-\omega_0)\, \underbrace{e^{i\tilde{k}(\omega)a}}_{\text{Phase shift in medium}}\; \underbrace{e^{i\frac{\omega}{c}(z-a)}}_{\text{in vacuum}}\, e^{-i\omega t}$$

$$\Rightarrow E^{(+)}(z,t) \cong \int_{-\infty}^{\infty} \frac{d\Delta}{2\pi}\, \tilde{\mathcal{E}}(\Delta)\, e^{i\tilde{k}(\omega_0+\Delta)a}\, e^{-i\Delta\left(t-\frac{z-a}{c}\right)}\, e^{i\omega_0\left(\frac{z-a}{c}-t\right)}$$

Again, $\tilde{k}(\omega_0+\Delta) \approx \tilde{k}(\omega_0) + \frac{\Delta}{v_g} = \tilde{n}(\omega_0)\frac{\omega_0}{c} + \frac{\Delta}{v_g}$

$$\Rightarrow \boxed{E^{(+)}(z,t) = E_0\, \underbrace{e^{i\frac{\omega_0}{c}a(\tilde{n}(\omega_0)-1)}}_{\text{Gain factor} \approx 1}\; \underbrace{e^{i\omega_0\left(\frac{z}{c}-t\right)}}_{\text{Carrier wave propagation}}}$$

$z>a$

$$\underbrace{\int_{-\infty}^{\infty} \frac{d\Delta}{2\pi}\, \tilde{\mathcal{E}}(\Delta)\, e^{-i\Delta\left(t-\frac{z-a}{c}-\frac{a}{v_g}\right)}}_{E_0 \exp\left\{-\frac{\left[\left(t-\frac{a}{v_g}\right)-\left(\frac{z-a}{c}\right)\right]^2}{2\tau^2}\right\}}$$

@ $z=a$ $\quad E^{(+)}(z,t) = \mathcal{E}_0 \, \underset{\substack{\uparrow \\ \text{phase}}}{e^{i\phi}} \, e^{-\frac{(t-\frac{a}{v_g})^2}{2\tau^2}}$

peak @ $t = \frac{a}{v_g}$

If $v_g<0$, peak emerges before peak even enters the medium!

(d) In all of these expressions, we neglected group-velocity dispersion. This will be true when

$$\tau^2 \gg \left|\frac{d^2k(\omega)}{d\omega^2}\right|_{\omega_0} a = \frac{a}{c} \left.\frac{d^2(\omega n)}{d\omega^2}\right|_{\omega_0}$$

The G.V.D. coeff. $\left|\frac{d^2}{d\omega^2}(\omega n)\right|_{\omega_0} = \left|\frac{d}{d\omega} v_g\right|_{\omega_0}$

$$\approx 24 \frac{\omega_p^2 \gamma}{\Delta^4} \quad \text{for } \gamma \ll \Delta$$

$$\Rightarrow \boxed{\frac{a}{c\tau} \ll \frac{1}{24} \frac{\Delta^4 \tau}{\omega_p^2 \gamma}}$$