Physics 511
Electrodynamics

Problem Set #1: DUE Thurs. Jan. 30, 2025

Problem 1. Vector identities (15 points)

Use tensor notation and the Einstein summation convention to prove the following:

(a) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$
(b) $\nabla\times(\psi\mathbf{A})=\psi(\nabla\times\mathbf{A})-(\mathbf{A}\times\nabla)\psi$
(c) $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

For part (c), show this the "old fashion way", by explicitly expressing both sides of the equation in Cartesian coordinates (i. e., take the curl, gradient, and divergences)

Problem 2. Irreducible tensors (10 points)

Generally, a tensor in Cartesian coordinates of rank K can be decomposed into a sum of so called *irreducible* tensors which transform in a simple way under rotations. As we will see, the components of ther irreducible tensor are closely realted to the components of the spherical harmonics.

For example, a rank 2 Cartesian tensor U_{ij} can be decomposed as

$$U_{ij}=U_{ij}^{(0)}+U_{ij}^{(1)}+U_{ij}^{(2)}$$

where $U_{ij}^{(0)}=\frac{\mathrm{Tr}(U)}{3}\delta_{ij}$, is the "scalar" part with 1 component.

$U_{ij}^{(1)}=\frac{1}{2}\left(U_{ij}-U_{ji}\right)$, is the "vector" part with 3 independent components.

$U_{ij}^{(2)}=\frac{1}{2}\left(U_{ij}+U_{ji}\right)-\frac{\mathrm{Tr}(U)}{3}\delta_{ij}$, is the "tensor part" with 5 independent components

In general, the lth irreducible tensor will have $2l+1$ independent components.
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(a) For a tensor formed from the outer product of two vectors, $U_{ij} \equiv A_i B_j$, show that

$$U_{ij}^{(0)} = \frac{\mathbf{A}\cdot\mathbf{B}}{3}\delta_{ij} \qquad U_{ij}^{(1)} = \frac{1}{2}\varepsilon_{ijk}\left(\mathbf{A}\times\mathbf{B}\right)_k \qquad U_{ij}^{(2)} = \frac{A_i B_j + A_j B_i}{2} - \frac{\mathbf{A}\cdot\mathbf{B}}{3}\delta_{ij}.$$

(b) Show that the "contraction" of two Caretsian tensors decompose as

$$U_{ij}W_{ij} = Tr\left(U W^T\right) = U_{ij}^{(0)}W_{ij}^{(0)} + U_{ij}^{(1)}W_{ij}^{(1)} + U_{ij}^{(2)}W_{ij}^{(2)}$$

This is a kind of projection between "orthogonal spaces".

<u>Problem 3.</u> Helmholtz's theorem (10 points)

Any vector field can be decomposed into $\mathbf{V}(\mathbf{x}) = \mathbf{V}_T(\mathbf{x}) + \mathbf{V}_L(\mathbf{x})$, where: $\mathbf{V}_T(\mathbf{x})$ is the "transverse" or "solenoidal" component, satisfying $\nabla\cdot\mathbf{V}_T(\mathbf{x}) = 0$, and $\mathbf{V}_L(\mathbf{x})$ is the "longitudinal" or "irrotational" component, satisfying $\nabla\times\mathbf{V}_L(\mathbf{x}) = 0$.
Vector calculus then implies $\mathbf{V}_T(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x})$, $\mathbf{V}_L(\mathbf{x}) = -\nabla\phi(\mathbf{x})$, where $\mathbf{A}$ and ϕ are the "vector potential" and "scalar potential". According to Helmholtz's theorem,

$$\phi(\mathbf{x}) = \frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}', \qquad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int\frac{\nabla'\times\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

where ∇' means that the derivatives are with respect to the "primed variables"

Prove it! Hint: Consider $\nabla\times\left(\nabla\times\int\frac{\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)$

A corollary to this theorem is that the divergence and curl completely define the vector field.