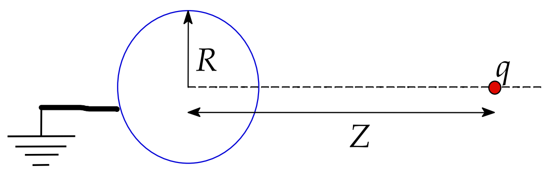
Physics 511, Electrodynamics

Problem Set #3

Due. Thursday Feb. 20, 2025

Problem 1 (20 points): Boundary-valued problem

Consider a point charge q, a distance Z from the center of a grounded sphere at the origin (take q on the z-axis).



(a) Using the method of images, show that the electrostatic potential outside the sphere is (in spherical coordinates)

$$\phi(r,\theta)= \frac{q}{\sqrt{r^2+Z^2-2rZ\cos\theta}} - \frac{q}{\sqrt{\left(\frac{rZ}{R}\right)^2+R^2-2rZ\cos\theta}}$$

(b) Show that the surface charge density induced on the surface of the sphere is

$$\sigma(\theta) = -\frac{q}{4\pi R}\frac{Z^2-R^2}{\left(R^2+Z^2-2RZ\cos\theta\right)^{3/2}}$$

and the total induced charge is -qR/Z, going to zero when q is infinitely far from sphere.

(c) With Dirichelet boundary conditions, show that the Green's function is

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{R^2}+R^2-2rr'\cos\gamma}}$$

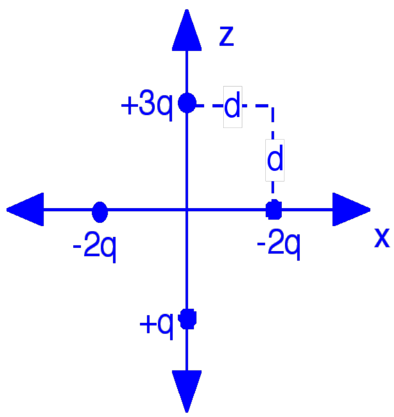
where $r=|\mathbf{x}|,\ r'=|\mathbf{x}'|\quad \cos\gamma = \hat{r}\cdot\hat{r}' = \cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$

(d) From the Green's function, with boundary condition of a potential on the surface given by $\Phi(R,\theta,\phi)$, show that the solution to Laplace's equation outside the sphere is

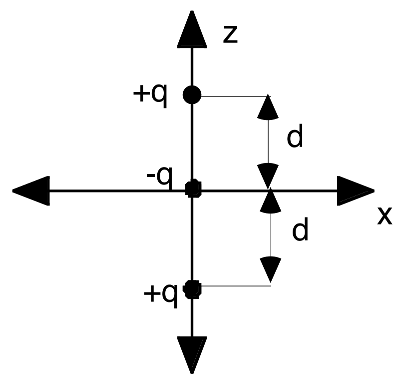
$$\Phi(R,\theta,\phi) = \frac{1}{4\pi}\int \Phi(R,\theta',\phi')\frac{R(r^2-R^2)}{(r^2+R^2-2Rr\cos\gamma)^{3/2}}d\Omega'$$

Problem 2 (15 points)

(a) For each the charge distributions drawn below, explicitly calculate the first three (monopole, dipole, quadrupole) Cartesian and Spherical multipole moments (all components).



(i)



(ii)

(b) Using these Cartesian multipole expansion, write the potential as a function of x,y, and z to order $(d/r)^3$

(c) Write the exact potential and expand to the same order to check your result.

(d) Extra credit (5 points). Plot the equipotential contours in the x-z plane for the exact potential and the approximate potential found in part (b) (set q=d=1). Plot for regions for r/d~1 and r>>d. Comment on these plots.

Problem 3: A Jackson Problem! (15 points)

4.6 A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

(*a*) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4} Q \left(\frac{\partial E_z}{\partial z}\right)_0$$

(*b*) If it is known that $Q = 2 \times 10^{-24}\ \mathrm{cm}^2$ and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of e/a_0^3, where $a_0 = \hbar^2/me^2 = 0.529 \times 10^{-8}$ cm is the Bohr radius in hydrogen.

(*c*) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153} $(Z=63)$ has a quadrupole moment $Q = 2.5 \times 10^{-24}\ \mathrm{cm}^2$ and a mean radius

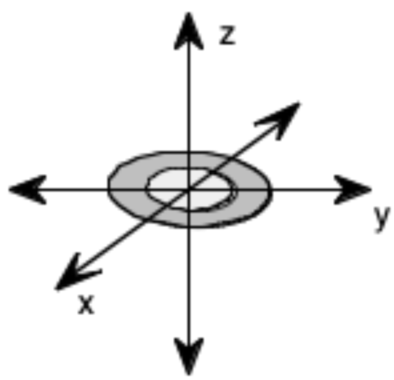
$$R = (a+b)/2 = 7 \times 10^{-13}\ \mathrm{cm}$$

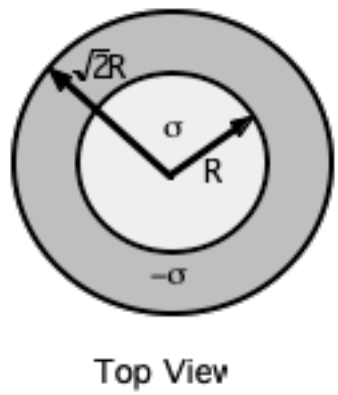
determine the fractional difference in radius $(a-b)/R$.

Problem 4 (15 points)

Multipole moments for an azimuthally symmetric charge distribution.

Consider a disk of radius R and surface charge/area σ surrounded by an annulus of charge with outer radius $\sqrt{2}R$, inner radius R, and surface charge/area –σ as sketched below.





(a) Find the multipole expansion of the potential up to the quadrapole moment. Express your final answer for the potential in spherical coordinates in terms of Legendre polynomials.

(b) Use direct integration to show that on the z-axis, the potential is

$$\Phi(z) = 2\pi\sigma\left(2\sqrt{z^2+R^2} - \sqrt{z^2+2R^2} - z\right)$$

(c) Check your result. Since the charge distribution is azimuthally symmetric, outside the charge distribution the potential satifies Laplace's equation as given in in the form,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

For r > $\sqrt{2}$ R use the answer to part (b) as a boundary condition at θ=0, (i.e. $\Phi(r,\theta=0) = 2\pi\sigma\left(2\sqrt{r^2+R^2} - \sqrt{r^2+2R^2} - r\right)$), together with the b.c. $\lim_{r\to\infty}\Phi(r,\theta)\to 0$, to find the expansion coefficients, $\{A_l, B_l\}$ up to order (R/r)^3.