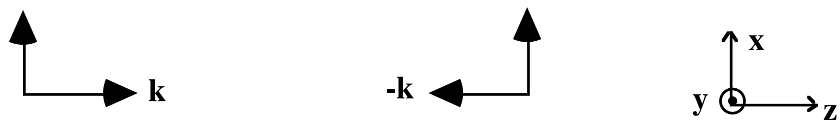
Physics 511: Electrodynamics
Problem Set #6
Due Thursday March 27

(1) Standing Waves (20 points)

Consider the superposition of two counterpropagating and plane waves with the same frequency, whose electric fields have the same polarization

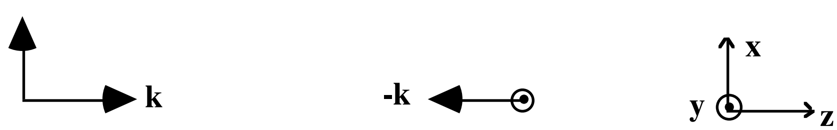


$\mathbf{E}_1 = \hat{\mathbf{x}}\, E_0 \cos(kz - \omega t)$ $\mathbf{E}_2 = \hat{\mathbf{x}}\, E_0 \cos(kz + \omega t)$

(a) What is the total electric field $\mathbf{E}_3 = \mathbf{E}_1 + \mathbf{E}_2$? What is the total magnetic field, $\mathbf{B}_3$? Sketch $|\mathbf{E}_3(z,t)|$ and $|\mathbf{B}_3(z,t)|$ as a function of z over one wavelength for $t = 0, T/4, T/2, 3T/4, T$, where T is the period of oscillation.

(b) What are the electric and magnetic energy densities as a function of z and t., and time averaged? What is the time average energy flux (intensity). Explain your answer.

Now consider now two the superposition to two counterpropagating traveling planes waves that are cross-polarized and 90∞ out of phase:



$\mathbf{E}_1 = \hat{\mathbf{x}}\, E_0 \cos(kz - \omega t)$ $\mathbf{E}_2 = \hat{\mathbf{y}}\, E_0 \sin(kz + \omega t)$

(c) The polarization of this wave is *not* inform in space. Show that, relative to the +z-direction the polarization varies over one half wavelength from positive-helicity circular (z=0), to linear (z=λ/8), to negative-helicity circular (z=λ/4), to linear (z=3λ/8), and back to positive-helicity circular (z=λ/2).

(d) Show that the total field can be written as a superposition of a positive-helicity circular, and a negative-helicity circular polarized *standing wave*. You should find that the nodes of one standing wave corresponds to the antinodes of the other.

(e) What is the intensity as a function of position?

Problem 2: Spherical Waves (20 points)

Consider the wave equation in three dimensions for a *scalar* field $\psi(\mathbf{r},t)$.

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 .$$

We seek solutions for monochromatic wave in spherical coordinates, independent of θ and ϕ - this corresponds, e.g., to waves generated by a point source.

(a) Let $\psi(\mathbf{r},t) = \tilde{\psi}(r)e^{-i\omega t}$. Show that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\psi}}{\partial r}\right) + k^2\tilde{\psi} = 0$, where $k = \omega / v$.

Show that the most general solution can be written as superpositions of

$$\psi_1(\mathbf{r},t) = u_0\frac{\cos(kr - \omega t - \phi)}{r} \quad \text{and} \quad \psi_2(\mathbf{r},t) = u_0\frac{\cos(kr + \omega t - \phi)}{r} .$$

What is the physical difference between ψ_1 and ψ_2 ?

(b) In free space we know that Maxwell's Equations imply that the waves are transverse. A first guess at the vector spherical wave would choose the polarization in the $\hat{\phi}$ of $\hat{\theta}$ direction. Show that $\mathbf{E} = E_0\frac{\cos(kr - \omega t)}{kr}\hat{\phi}$ does not satisfy Maxwell's Equations.

(c) The simplest possible *vector* spherical wave for the electric field in free space is

$$\mathbf{E}(r,\theta,\phi) = E_0\left(\frac{\sin\theta}{kr}\right)\left[\cos\left(kr - \omega t\right) - \left(\frac{1}{kr}\right)\sin(kr - \omega t)\right]\hat{\phi} .$$

Show that $\mathbf{E}$ obeys all four Maxwell's equation, in vacuum, and find the associated magnetic field.

(d) Show that in the limit $kr \ll 1$, the magnetic field has the *instantaneous* form of a static dipole field $\mathbf{B}(\mathbf{x},t) = \frac{3(\mathbf{m}(t)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}(t)}{r^3}$, where $\mathbf{m}(t) = \frac{E_0}{k^3}\cos\omega t\,\hat{\mathbf{e}}_z$.

(e) Find the time average Poynting vector; does it point in the expected directions and have the expected fall off with *r*.

(f) Find the flux of energy through a sphere, radius R, centered at the origin, and comment on your result.

Problem 3: Angular momentum in electromagnetic waves (Jackson 3rd Edition: Problem 7.27). (EXTRA CREDIT 15 points)

The angular momentum of the electromagnetic field is $\mathbf{L} = \frac{1}{4\pi c}\int d^3x\, \mathbf{x}\times\left(\mathbf{E}\times\mathbf{B}\right)$, where the integration is over all space.

(a) Eliminate the magnetic field in favor of the vector potential. Show that for field localized to a finite region of space,

$$\mathbf{L} = \frac{1}{4\pi c}\int d^3x\left[\mathbf{E}\times\mathbf{A} + E_i\left(\mathbf{x}\times\nabla\right)A_i\right] \text{ (sum over components } i \text{ implicit).}$$

The second term is referent to as the "orbital" angular momentum because of the presence of the orbital angular momentum operator familiar in wave mechanics, $\mathbf{L}_{op} = -i\left(\mathbf{x}\times\nabla\right)$. The first term relates to the vector nature of the filed itself as is referred to as the "spin" angular momentum.

(b) Consider a Fourier decomposition of the vector potential into transverse, circularly polarized plane waves

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mu=\pm}\int\frac{d^3k}{(2\pi)^3}\left[\tilde{A}_\mu(\mathbf{k})\,\mathbf{e}_\mu(\hat{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + c.c.\right],$$

where $\mathbf{e}_\pm(\hat{\mathbf{k}})$ are circular polarizations orthogonal to $\hat{\mathbf{k}}$.

Show that the *time average* of the "spin" angular momentum is

$$\left\langle \mathbf{L}_{spin}\right\rangle = \frac{1}{2\pi c}\int\frac{d^3k}{(2\pi)^3}\,\mathbf{k}\left[\left|\tilde{A}_+(\mathbf{k})\right|^2 - \left|\tilde{A}_-(\mathbf{k})\right|^2\right].$$

(c) Calculate the total energy in the field using the expansion in (b). If we associate, according to quantum mechanics, an energy density of $\frac{\hbar\omega}{V}$ per mode (i.e., Fourier component), what is the "spin" angular momentum of a positive or negative helicity electromagnetic mode, both in direction and magnitude? These are the fundamental characteristics of "photons".