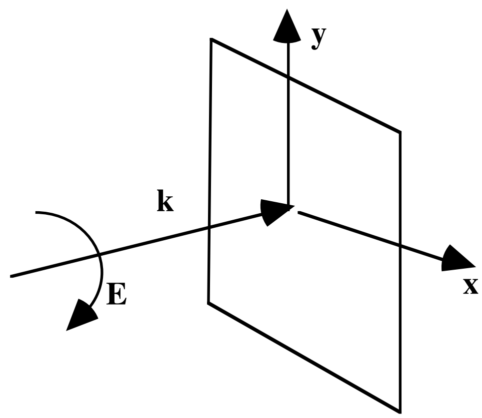
Physics 511

Electrodynamics

Problem Set #7: DUE Friday April 3, 2025

Problem 1: (Spin) Angular momentum in electromagnetic plane waves (15 Points)

Consider a positive helicity, monochromatic plane wave incident on a completely absorbing material.



Model the motion of the charges in the material as damped simple harmonic oscillators, driven by the incident electric field.

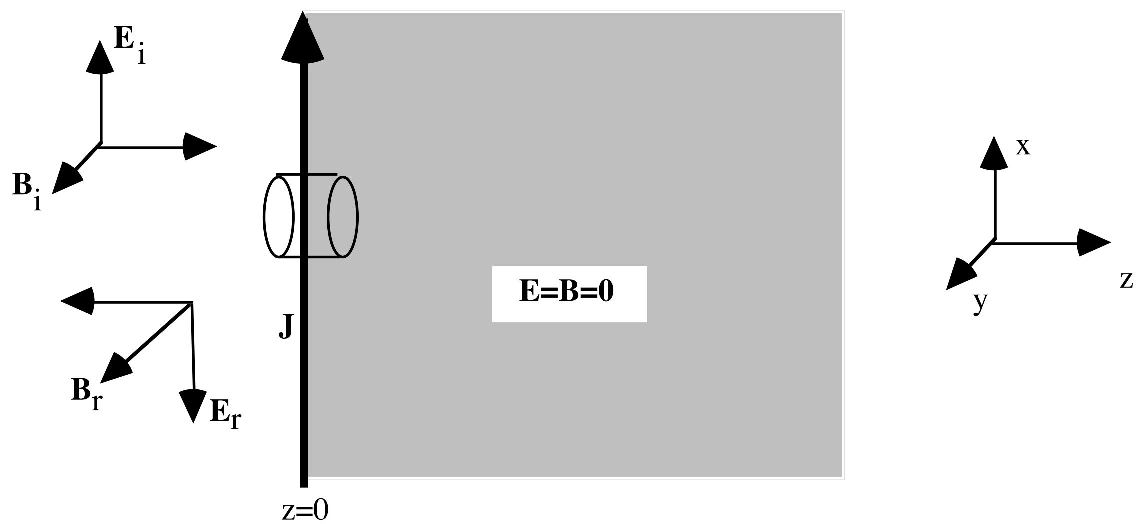
(a) Find the steady state, time average torque on the charges.
(b) Find the time average rate at which the field does work on the charges.
(c) Use these to deduce that the average angular momentum in the field is related to the average energy in the field by,

$$\langle \mathbf{L} \rangle = \frac{\langle W \rangle}{\omega} \hat{\mathbf{k}}$$

Please comment on this result.

Problem 2: Radiation pressure due to plane waves (10 points)

We know that electromagnetic waves carry momentum as discussed in class. When incident on an absorbing/reflecting medium, this exerts a pressure. Consider a plane electromagnetic wave is normally incident on a perfectly conducting mirror (see sketch). Currents flow in the surface of the mirror (z=0), generating the reflected wave.



The incident and reflected electromagnetic fields have the form ($z<0$)

$$\text{incident}\quad \begin{cases} \mathbf{E}_i = \mathbf{e}_x E_0 \cos(\omega t - kz) \\ \mathbf{B}_i = \mathbf{e}_y E_0 \cos(\omega t - kz) \end{cases}$$

$$\text{reflected}\quad \begin{cases} \mathbf{E}_r = -\mathbf{e}_x E_0 \cos(\omega t + kz) \\ \mathbf{B}_r = \mathbf{e}_y E_0 \cos(\omega t + kz) \end{cases}$$

For $z>0$, the fields are identically zero.

(a) Calculate the time-averaged radiation pressure on the mirror by integrating the Maxwell stress-tensor over the closed surface of the "pill box" shown above.

(b) Obtain the same result by integrating $\frac{1}{c}\mathbf{J}\times\mathbf{B}$ over the volume of the pill box. You will first need to calculate the surface current flowing at z=0. Be careful to disregard the force on the current sheet due to its own electromagnetic fields (i.e., the "self-energy").

Problem 3. Chromatic dispersion in conductors. (20 Points)

In discussing wave propagation in conductors, we derived the dispersion relation

$$k^2 = \frac{\omega^2}{c^2}\mu\varepsilon\left(1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right),$$

but neglected the frequency response of the conductivity σ. A simple model (analogous to the Lorentz oscillator model for dielectrics) which accounts for the frequency response of the conductor, treats electrons as a free gas. The result is the *Drude model.*, which gives a frequency dependent complex conductivity according to $\tilde{\mathbf{J}}(\omega) = \tilde{\sigma}(\omega)\tilde{\mathbf{E}}(\omega)$

(a) Show that in the Drude model the current density satisfies the generalized Ohm's law, $\frac{d\mathbf{J}}{dt} + \nu_e \mathbf{J} = \frac{\omega_p^2}{4\pi}\mathbf{E}$, and from this derive the complex conductivity, $\tilde{\sigma}(\omega) = \frac{\omega_p^2/4\pi}{\nu_e - i\omega}$, where ν_e is the electron collision rate and $\omega_p = \sqrt{4\pi N_e e^2/m_e}$ is the plasma frequency with electron density N_e.

(b) Let us assume the inequality $\omega_p >> \nu_e$ (typical for a good conductor). There are three regimes in which the conductor behaves very differently:
(i) Low freq.: $\omega << \nu_e$, (ii) Intermediate: $\nu_e << \omega < \omega_p$ (iii) High freq.: $\omega > \omega_p$
Show that in regime (i) the metal responds essentially as a DC conductor, whereas in regimes (ii) and (iii), the metal looks like a plasma with dispersion relation $c^2k^2 = \omega^2 - \omega_p^2$ (taking ε=μ=1). Explain this *physically*.

(c) Show that in regime (ii) the conductor is *perfectly* reflecting, whereas in regime (iii) the metal is transparent with effective index of refraction $n(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$. What are the phase and group velocities of wave propagation in regime (iii)?

(d) Given the empirical DC conductivity of silver as $\sigma = 6.17\times10^7$/(ohm m) and your knowledge of the periodic table, estimate the plasma frequency and collision rate of electrons for silver at standard temperature and pressure.

(e) Give an approximate expression for the real and imaginary parts of the index of refraction at a typical microwave, visible, and x-ray frequency (take ε=1, μ=1). What is the skin depth of silver at a radio frequency?