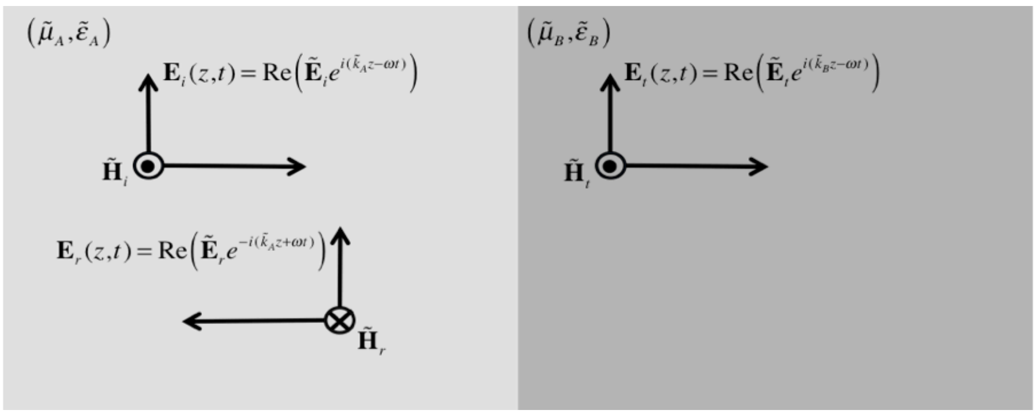
Physics 511

Electrodynamics

Problem Set #8: Due Friday April 11, 2025

Problem 1: Reflection and Transmission (25 points)

Consider two semi-infinite homogeneous media characterized by complex permeability and permittivity $\tilde{\mu}_A\,\tilde{\varepsilon}_A$ and $\tilde{\mu}_B\,\tilde{\varepsilon}_B$ respectively, with the interface at z=0. A plane wave propagating in medium A will be partially transmitted and partially reflected as sketched.



(a) With the directions of **E** and **H** as shown, define the amplitude reflection and transmission coefficients $\tilde{r}=\tilde{E}_r/\tilde{E}_i$ and $\tilde{t}=\tilde{E}_t/\tilde{E}_i$. Using the boundary conditions at z=0, show that,

$$\tilde{r}=\frac{\tilde{Z}_B-\tilde{Z}_A}{\tilde{Z}_A+\tilde{Z}_B}, \qquad \tilde{t}=\frac{2\tilde{Z}_B}{\tilde{Z}_A+\tilde{Z}_B},$$

where $\tilde{Z}=\sqrt{\tilde{\mu}/\tilde{\varepsilon}}$ is the wave impedance. Reflection is thus due to an impedance mismatch between the two media.

(b) For nonmagnetic, nonabsorbing dielectrics show that

$$r=\frac{n_A-n_B}{n_A+n_B}, \qquad t=\frac{2n_A}{n_A+n_B}$$

where $n=\sqrt{\varepsilon}$ is the index of refraction.

(c) Define the energy transmission and reflection coefficients, $R = \frac{I_r}{I_t}$, $T = \frac{I_t}{I_i}$, where I is the wave intensity. For nonabsorbing media A and B, find these coefficients and show that $R + T = 1$, as is required by energy conservation.

(d) Now include absorption through the imaginary part of the dielectric constant. If the wave travels from vacuum to a dispersive dielectric defined by $\tilde{\varepsilon}(\omega)$, show that

$$R = \left| \frac{1 - \tilde{n}(\omega)}{1 + \tilde{n}(\omega)} \right|^2 \text{ and } T = \frac{4\,\mathrm{Re}\left(\tilde{n}(\omega)\right)}{\left|1 + \tilde{n}(\omega)\right|^2}$$

where $\tilde{n}(\omega) = \sqrt{\tilde{\varepsilon}(\omega)}$ is the complex index of refraction.

(e) Consider the case when medium A is the vacuum and medium B is a nonmagentic highly conducting metal with complex permittivity $\tilde{\varepsilon}(\omega) \approx i\frac{4\pi\tilde{\sigma}(\omega)}{\omega}$. Suppose a low frequency wave is incident on the conductor so that $\tilde{\sigma}(\omega)$ can be approximated by its DC value $\sigma_0 >> \omega$. Show that the energy reflection coefficient is

$$R \approx 1 - \sqrt{\frac{2\omega}{\pi\sigma_0}}$$

For silver with DC conductivity $\sigma_0 = 6.14 \times 10^7$ ohm m^{-1} (SI units), what fraction of the energy of a wave of free space wavelength $\lambda = 25\,\mu m$ is absorbed?

Problem 2: Group-velocity dispersion (GVD). (25 Points)

We seek the equation of motion for the envelope of a quasimonchromatic pulse propagating in a dispersive transparent medium.

(a) Begin with the Helmholtz equation for the positive frequency Fourier component of the electric field, $\left(\nabla^2 + k^2(\omega)\right)\tilde{\mathbf{E}}^{(+)}(\mathbf{x},\omega) = 0$, where $k(\omega) = n(\omega)\omega / c$. Assume one-dimensional propagation of a polarized beam (ignore transverse effects) and make the slowly varying envelope approximation (SVEA) via the ansatz

$$\tilde{\mathbf{E}}^{(+)}(\mathbf{x},\omega) = \hat{\mathbf{e}}\tilde{\mathcal{E}}(z,\omega-\omega_0)e^{ik_0 z}$$

where $k_0 = n(\omega_0)\omega_0 / c$ and $\left|\frac{\partial\tilde{\mathcal{E}}}{\partial z}\right| << k_0\left|\tilde{\mathcal{E}}\right|$. Under the assumption that $\tilde{\mathcal{E}}^{(+)}$ is narrowly peaked at $\omega - \omega_0 = 0$, show that up to GVD terms

$$i\frac{\partial\tilde{\mathcal{E}}(z,\Delta)}{\partial z} + \left(\frac{1}{v_g}\Delta + \frac{k_0''}{2}\Delta^2\right)\tilde{\mathcal{E}}(z,\Delta) = 0$$

where $\Delta = \omega - \omega_0$, $\frac{1}{v_g} = \left.\frac{dk}{d\omega}\right|_{\omega_0}$, $k_0'' = \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}$, and we have assumed $\left|k_0 k_0''\right| >> \left|k_0'\right|^2$.

(b) Take the inverse Fourier transform to show that the pulse envelope satisfies the propagation equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t) = -i\frac{k_0''}{2}\frac{\partial^2\mathcal{E}}{\partial t^2}.$$

(c) Change variables from (t,z) to (τ,z) where $\tau = t - z/v_g$ is the "retard time". Let $A(\tau,z) = \mathcal{E}(z,t = \tau + z/v_g)$, and show that

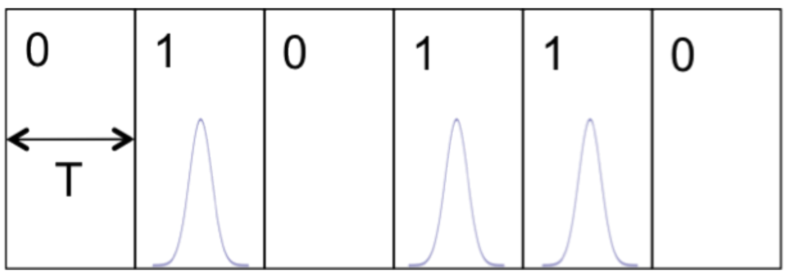
$$\frac{\partial A(\tau,z)}{\partial z} = -i\frac{k_0''}{2}\frac{\partial^2 A}{\partial\tau^2}$$

Show that this is isomorphic to the time-dependent Schrödinger for a free particle moving in one dimension with *z* as the independent variable and τ as "position". From this, deduce how a Gaussian wave packet will spread due to GVD.

Dispersion and the resulting spreading of a wave packet is a serious problem in optical communications, where pulses of light are sent through optical fibers over very long distances. Optical fibers are basically wires of silica glass. The minimum absorption (due to scattering from imperfections and other loss mechanisms) is at a wavelength of λ=1.55 μm. (The power) attenuation coefficient at this frequency is found to be $2k_I=2\times10^{-5}$ cm^{-1}, and the dispersion coefficient $\dfrac{d^2k}{d\omega^2}$ is about –25 ps^2/km. (ps = picosecond)

(d) Estimate the characteristic distance of spreading of a 20 ps pulse at this wavelength. How does it compare with the attenuation length?

(e) One way to send a digital signal in an optical fiber is divide up time into "windows" of duration T. If a pulse appears in the window, the bit is a "1" - no pulse, the bit is a "0".



The rate of data transmission (bits/second) is then $1/T$. The window is taken in order to avoid overlap. It can be shown that if we can tolerate a bit rate error no greater than 10^{-9}, then the time window must be 2 times the pulse duration. Of course the name of the game is to maximize the bit-rate over the longest possible distances. However, very short pulses have broad spectra, and thus spread very rapidly.

What is the maximum distance one could send 20 ps pulses at 10 Gbit/sec (in a 100 ps time window) before dispersion makes the bit error-rate intolerable. Is absorption a problem?

Problem 3: Negative group velocity. (25 points Extra credit)

Recall the group velocity $v_g = \left(\frac{dk_R}{d\omega}\right)^{-1} = \frac{c}{n_R(\omega)+\omega\, dn_R/d\omega}$, where we have written the real part of the wave number in terms of the real index of refraction $k_R(\omega) = \frac{\omega}{c} n_R(\omega)$.

The "group index" is defined $n_g(\omega) = \frac{c}{v_g} = \frac{d(\omega n_R)}{d\omega} = n_R(\omega) + \omega \frac{dn_R}{d\omega}$. When $\frac{dn_R}{d\omega} < 0$ we are in a region of "anomalous dispersion" and v_g can begreater that *c* or even negative!

As discussed in Jackson, if $\mathrm{Im}(\tilde{\chi}(\omega)) > 0, \forall\omega$, and we are in a transparent band, we must have $n_g > 1$, and the group velocity is

subluminal. Thus for a "passive medium", $\mathrm{Im}(\tilde{\chi}(\omega)) > 0$, we cannot have both transparency and anomalous dispersion. On the other hand, in active medium with

$\mathrm{Im}(\tilde{\chi}(\omega)) < 0$ for some ω, all bets are off. In a seminal experiment in 2000, Wang, Kuzmich and Dogariu demonstrated such negative group velocities based on a 1994 proposal of Chiao and Steinberg for dispersion between two "gain lines". The purpose of this problem is to work through the details of how this works.

Consider two spectral lines at resonance frequencies $\omega_{1,2}$ and an oscillator strength of -1, i.e. these are amplifiers rather than absorbers. Physically this is achieved by population inversion in an atomic gas. We take each resonance to have a linewidth γ. The complex index of refraction, using the Lorentz oscillator model is then,

$$\tilde{n}(\omega) = \sqrt{1+4\pi N\tilde{\alpha}(\omega)} = 1 + 2\pi N\tilde{\alpha}(\omega) = 1 - \frac{\omega_p^2}{\omega_1^2 - \omega^2 - i\gamma\omega} - \frac{\omega_p^2}{\omega_2^2 - \omega^2 - i\gamma\omega}.$$

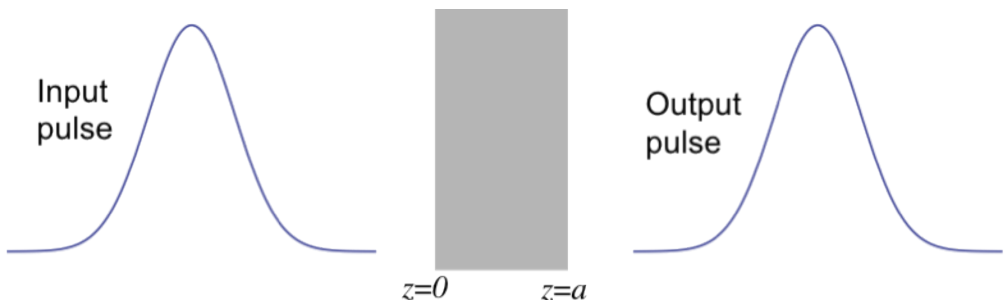
(a) Defining $\omega_{1,2} = \omega_0 \pm \Delta/2$, plot $\mathrm{Re}(\tilde{n}(\omega)-1)$ and $\mathrm{Im}(\tilde{n}(\omega))$ for the parameters $\omega_0/2\pi = 352$ THz, $\Delta/2\pi = 1.8$ MHz, $\gamma/2\pi = 720$ kHz, and $\omega_0 = 21.6$ MHz. Use as your plot domain $\omega_0 - 2\Delta < \omega < \omega + 2\Delta$. Note that between the resonance lines we are in a region of anomalous dispersion, but the medium is essentially transparent, with a small amount of gain.

(b) For $\gamma, \Delta << \omega_0$ show that $n_g = \left.\frac{d(\omega n_R(\omega))}{d\omega}\right|_{\omega_0} \approx 1 - \frac{2\omega_p^2(\Delta^2-\gamma^2)}{(\Delta^2+\gamma^2)^2}$, and from this show that to have negative group velocities we must have $\Delta \le \sqrt{2}\,\omega_P$. The region of interest is $\gamma << \Delta << \omega_0$, as for this show that at ω_0 the group velocity is

$$v_g \approx -\frac{c}{2}\frac{\Delta^2}{\omega_p^2}.$$

In the Wang experiment $\Delta/\omega_p \approx 1/12$, and $v_g \approx -c/310$.

(c) Consider now a Gaussian pulse incident on the gain medium in the region $0 < z < a$. For $z < 0$ and $z > a$ we have vacuum. The incident wave is a quasimonochromatic Gaussian pulse of width τ, with peak amplitude E_0 and carrier frequency ω_0.



Show that up to first order dispersion (ignoring the group velocity dispersion)

$$E(z,t) = \begin{cases} E_0 e^{-(z/c-t)^2/2\tau^2} e^{i\omega_0(z/c-t)} & \textit{for } z<0 \\ E_0 e^{-(z/v_g-t)^2/2\tau^2} e^{i\omega_0(\tilde{n}(\omega_0)z/c-t)} & \textit{for } 0<z<a \\ E_0 e^{i\omega_0(\tilde{n}(\omega_0)a-1)/c} e^{-((z-a)/c-(t-a/v_g))^2/2\tau^2} e^{i\omega_0(z/c-t)} & \textit{for } z>a \end{cases}$$

The factor $e^{i\omega_0(\tilde{n}(\omega_0)a-1)/c} = e^{\omega_p^2\gamma a/\Delta^2 c}$ is the gain, and is close to 1 for the parameters here.

The peak of the Gaussian pulse emerges from $z = a$ at the time $t_g = \dfrac{a}{v_g}$.

Thus if $v_g < 0$, the peak exits the medium before it enters at z=0!

This surprising behavior arises because the smooth tail of the Gaussian contains all the information about the arrival time of the peak. No information travels faster than the speed of light!

(d) Show that in order to satisfy no spreading of the wave packet we must have

$$\frac{a}{c\tau} << \frac{1}{24}\frac{\Delta^4\tau}{\omega_p^2\gamma}.$$

Under these conditions the pulse emerges essentially unchanged in shape, just slightly advanced. In the Wang et al experiment the was well satisfied with $a = 6\,cm$, and $c\tau = 300\ m$