Physics 511 Spring 2025

Problem Set #10: Extra Credit
Due Friday May 8

Problem 1. Scattering in one dimension (20 points)

Consider a system in which all variation in the material properties vary only in one direction (call it the *z* direction); e.g. a infinite slab of dielectric, homogeneous along the transverse dimension. One canshow that the formal solution for the electric field is:

$$E(z,t) = E_{inc}(z,t) - \frac{2\pi}{c}\int dz'\, J(z', t - |z - z'|/c),$$

where E_{inc} is the incident field and J is the current density. Note that the radiated field is proportional to the current, and is 180° out of phase with the incident radiated field

(a) For the case of a monochromatic incident plane wave, $E_0 e^{i(kz-\omega t)}$, and linear dielectric consisting of a smooth distribution of scatterers with linear polarizability α and density $N(z)$, show that we arrive at a formal integral equation solution for the complex amplitude,

$$E(z) = E_0 e^{ikz} + i2\pi\alpha k\int dz'\, e^{ik|z-z'|} N(z')E(z').$$

Note for α real, the scattered wave is 90° out of phase with the incident wave.

(b) Now suppose we have a very thin dielectric slab of thickness Δz and dilute particle density N_0. Show that in the first Born approximation, the forward scattered wave is,

$$E(z) = E_0 e^{ikz}(1 + i(2\pi N_0\alpha)k\Delta z) \approx E_0 e^{ikz} e^{i(n-1)k\Delta z},$$ where n is the index of refraction.

Interpret this important result!

(c) The first Born approximation can give unphysical results if we are not careful in interpreting them. Consider the case of a *nonabsorbing* dielectric sheet at the origin, with surface density N, so that the volume density is N(z)=Nδ(z). This can be though of as a beam splitter which partially transmits part of the wave and reflects the rest. Show that under the first Born approximation, energy is not conserved, i.e. if we define transmission and reflection coefficients as $t \equiv E_{trans} / E_0$, and $r \equiv E_{ref} / E_0$, $|t|^2 + |r|^2 \neq 1$

(d) For this special problem we can solve the integral equation *exactly* to all orders. Do it, and show that now energy is conserved. (*Hint*: The exact solution will have the form $E(0)e^{ik|z|}$).

Problem 2: Radiation reaction (20 points)

Consider an electron on a spring with resonance frequency ω0. Even without friction, the oscillator will experience a damping force as since it is loosing energy to electromagnetic radiation ("radiation resistance"). There will be a damping rate Γ

(a) Given some initial displacement A , derive an expression for the energy in the damped harmonic oscillator average of a period of oscillation. assume ω0>>Γ.

(b) Equating the rate of decay of the damped oscillator to the Larmor formula for the power radiated, show that the classical radiation damping rate is

$$\Gamma = \frac{2e^2\omega_0^2}{3mc^3} .$$

This is the classical picture of "spontaneous emission" for quantum mechanical oscillator.

Estimate numerically the "radiative lifetime" found for the case of an electron oscillating a typical optical frequency (i.e. a frequency of visible light)., and compare it to a typical radiative lifetime for an optical transition in a real atom such as sodium.

(c) The damping coefficient found in part (b) is a special case of the more general problem of *radiation reaction*. Consider a scattering problem in which an incident wave of frequency ω is *elastically* scattered (i.e. the scattered wave has the same frequency as the incident wave) by the electron on a frictionless spring. Since there are no other channels, all energy absorbed by the bound electron must be reradiated. By equating the absorbed to scattered power, show that in general the damping coefficient will be a function of frequency,

$$\Gamma = \frac{2}{3}\frac{e^2}{mc^3}\omega^2 .$$

(d) The resulting polarizability for the electron (including radiation damping) leads to unphysical **noncausal** solutions to the equation of motion. In particular, using the expressing in (c), show that in the absence of any binding or external forces, the equation of motion for a the charge is,

$$\ddot{\mathbf{x}} = \tau \dddot{\mathbf{x}}, \text{ where } \tau = \frac{2}{3}\frac{e^2}{mc^3},$$

with run away solution: $\mathbf{x}(t) = \mathbf{x}(0) + (\dot{\mathbf{x}}(0) - \tau \ddot{\mathbf{x}}(0))t - \tau^2 \ddot{\mathbf{x}}(0)(1 - \exp(t/\tau))$. This is known as the Abraham-Lorentz equation (see Jackson).

As stated in "Classical Field Theory", by F. E. Low,
"The contradiction between energy conservation and causality is a genuine difficulty of classical electromagnetic theory describing the interaction of electromagnetic fields with *point* particles. This problem (the unwanted pole) does not appear in relativistic quantum electrodynamics; however other problems do [i.e. divergence of Feynman diagrams]"