Multipole Moments in Electrostatics

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I. Introduction

When confronted with complicated systems in physics it is often very useful to break it up into pieces that are simpler, or at least easy to characterize. For example when analyzing a complicated signal, such as speech, it is useful to decompose the signal into a spectrum of pure sinusoidal tones. Since we know the signal with each pure sine wave, if we know how much of each frequency is present we know the signal.

We want to apply this kind of thinking studying the field associated with a given charge distribution. There are relatively few situation where we can solve exactly for the field everywhere in space. These are highly symmetric cases where we can use Gauss' law. For more complicated situations we have to resort to something else. The question we ask is, "what are there characteristics of the charge distribution which, once known, give us the field?". One characteristic we know right away - the total net charge q_{net}. We have seen that far away from the charge distribution, the field looks that of a point charge with charge q_{net}. But what if q_{net}=0, or what if we want to find corrections to the point charge approximation? The other quantities characterizing the charge distribution tell us *how the charge is distributed in space* about a given origin. These are known as the *moments of the charge distribution.*

Moments of a distribution are important in many fields of mathematics and physics. For example. Suppose we have a one dimensional function which specifies the number of people standing at certain positions on a line. We might write this as P(n)= number of people at the nth position. Important characteristics of the distribution are:

- The total number of all the people on line: $N = \sum_n P(n)$
- The average of the position of the people: $\langle x \rangle = \sum_n x(n)\, P(n)$
- The average spread of people from the middle of the line: $\langle x^2 \rangle = \sum_n x^2(n)\, P(n)$, etc.

In general, all moments $\langle x^m \rangle = \sum_n x^m(n)\, P(n)$ completely specify the distribution.

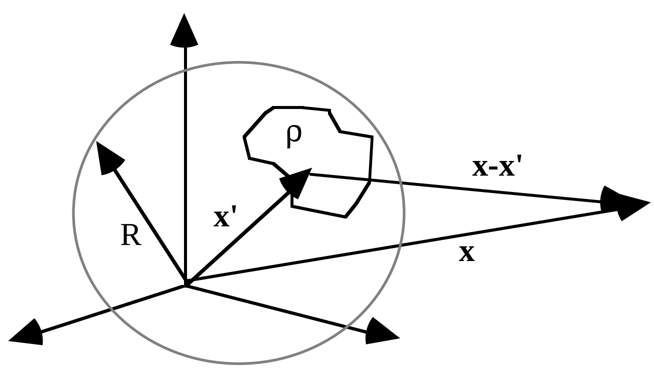
The same principles can be applied to characterize our charge distribution. Unlike the example about, we have both positive an negative charge (there are no "negative" people). In addition, we are working in three dimensions, which always makes life more difficult. But, the basic ideas are the same. We will characterize our charge distribution by:

- The total net charge:
- The average position of the positive charge along *x*, *y*, and *z*::
- The spreads of charge: $\langle x^2 \rangle, \langle y^2 \rangle, \langle z^2 \rangle, \langle xy \rangle, \langle xz \rangle, \langle yz \rangle$

These are respectively, the monopole moment, the dipole moment, and the quadrapole moment. Once we know these characteristics of the charge distribution, we can write down the potential as a superposition of known multipole potentials. Far enough away from the charge, the first nonvanishing moment dominates.

II. Formal Derivation of the Multipole Expansion of the Potential in Cartesian Coordinates

Consider a charge density $\rho(\mathbf{x})$ confined to a finite region of space (say within a sphere of radius R). For positions outside this region (r>>R), we seek an expansion of the exact potential in powers of r



$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{x}' \rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{V^{(0)}}{r} + \frac{V^{(1)}}{r^2} + \frac{V^{(2)}}{r^3} + \ldots \qquad (1)$$

This is known as the multipole expansion with

0th order: **Monopole potential** (falls off like 1/r)
1th order: **Dipole potential** (falls off like $1/r^2$)
2nd order: **Quadrapole potential** (falls off like $1/r^3$)
3rd order: **Octapole potential** (falls off like $1/r^4$)
etc....

In general, if the are far enough away from the charge distribution the first **nonvanishing** term in this expansion will dominate since higher order terms go to zero as a much larger power of r.

To determine this expansion explicitly consider the following:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\sqrt{r^2-2rr'\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'+r'^2}}\text{, where } \mathbf{x}=r\hat{\mathbf{r}}\text{ ,}\mathbf{x}'=r'\hat{\mathbf{r}}' \tag{2}$$

Now, we see an expansion in the small parameter (r'/r) (remember r determines the point of observation, and r' the positions of the charges). So we re express Eq. (2) as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{r}\left(1-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'(r'/r)+(r'/r)^2\right)^{-1/2}. \tag{3}$$

As an aside remember

$$(1+\delta)^n=1+n\delta+\frac{n(n-1)}{2}\delta^2+\ldots,\ \text{ for } \delta<1 \text{ (fewer terms needed the smaller } \delta\text{).}$$

So, if we let $n=-1/2$, and $\delta=-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'(r'/r)+(r'/r)^2 (<1,\ \text{since } r'/r<1)$, we have

$$\begin{aligned}
&\left(1-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'(r'/r)+(r'/r)^2\right)^{-1/2}\\
&=1-\frac{1}{2}\delta+\frac{3}{8}\delta^2+\ldots=1-\frac{1}{2}(-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'(r'/r)+(r'/r)^2)+\frac{3}{8}\left(-2\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'(r'/r)+(r'/r)^2\right)^2+\ldots\\
&\approx 1+\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\left(\frac{r'}{r}\right)-\frac{1}{2}\left(\frac{r'}{r}\right)^2+\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2\left(\frac{r'}{r}\right)^2\\
&=1+\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\left(\frac{r'}{r}\right)+\frac{1}{2}\left(3(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2-1\right)\left(\frac{r'}{r}\right)^2
\end{aligned}, \tag{4}$$

where we have kept term only to order $\left(r'/r\right)^2$. Plugging this into Eq. (3), we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{r} + \left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\frac{1}{r^2} + \frac{1}{2}\left(3\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)^2 - r'^2\right)\frac{1}{r^3}. \tag{5}$$

Note, here I have recombined $\mathbf{x}' = r'\hat{\mathbf{r}}'$. Plugging Eq. (5) into Eq. (1), remember that the integral is only over the positions of the charges (the primed variables). Therefore,

$$\begin{aligned}
\phi(\mathbf{x}) &= \int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \rho(\mathbf{x}') \\
&\approx \int d^3x' \left(\frac{1}{r} + \left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)\frac{1}{r^2} + \frac{1}{2}\left(3\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)^2 - r'^2\right)\frac{1}{r^3}\right)\rho(\mathbf{x}') \qquad . \\
&= \frac{\int d^3x'\ \rho(\mathbf{x}')}{r} + \frac{\hat{\mathbf{r}}\cdot\int d^3x'\ \mathbf{x}'\rho(\mathbf{x}')}{r^2} + \frac{1}{2}\frac{\int d^3x'\left(3\left(\hat{\mathbf{r}}\cdot\mathbf{x}'\right)^2 - r'^2\right)\rho(\mathbf{x}')}{r^3}
\end{aligned} \tag{6}$$

This is the expansion we sought in Eq. (1)! It is useful to factor out the dependence on the *observation position* and leave quantities that depend sole on the how the charge is distributed. We is easily done for the first two terms. We have,

$$\phi^{(0)}(\mathbf{x}) = \frac{q_{net}}{r}, \quad q_{net} = \int d^3x'\ \rho(\mathbf{x}')\ \text{(net charge)} \tag{7}$$

$$\phi^{(1)}(\mathbf{x}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \mathbf{p} = \int d^3x'\ \mathbf{x}'\rho(\mathbf{x}')\ \text{(dipole moment vector)} \tag{8}$$

The physical meaning of the terms will be described below.

In order to factor the dependence on the observation point out of the integral in V_2 we must resort to more sophisticated mathematics. The troublesome term can be written as,

$$(\hat{\mathbf{r}}\cdot\mathbf{x}')^2 = (\hat{\mathbf{r}}\cdot\mathbf{x}')(\mathbf{x}'\cdot\hat{\mathbf{r}}) = \hat{\mathbf{r}}\cdot(\mathbf{x}'\mathbf{x}')\cdot\hat{\mathbf{r}}$$

The expression $\mathbf{x}'\mathbf{x}'$ is the "outer product" of the vector $\mathbf{x}'$ with itself. It represents a 3×3 matrix with components $(\mathbf{x}'\mathbf{x}')_{ij} = x_i' x_j'$, where x_i' are the Cartesian components of the vector. Written out explicitly as a matrix

$$\mathbf{x}'\mathbf{x}' \doteq \begin{bmatrix} x'^2 & x'y' & x'z' \\ y'x' & y'^2 & y'z' \\ z'x' & z'y' & z'^2 \end{bmatrix}.$$

We can also write, $r'^2 = \hat{\mathbf{r}} \cdot (r'^2\, \overleftrightarrow{1}) \cdot \hat{\mathbf{r}}$, where $\overleftrightarrow{1}$ is the identity matrix, whose components are usually written as a "*Kronecker delta*",

$$r'^2\, \overleftrightarrow{1} \doteq \begin{bmatrix} r'^2 & 0 & 0 \\ 0 & r'^2 & 0 \\ 0 & 0 & r'^2 \end{bmatrix}.$$

Putting these together we have,

$$3(\mathbf{x} \cdot \mathbf{x}')^2 - r'^2 =$$
$$\begin{bmatrix} \hat{r}_x & \hat{r}_y & \hat{r}_z \end{bmatrix} \begin{bmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2 - r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2 - r'^2 \end{bmatrix} \begin{bmatrix} \hat{r}_x \\ \hat{r}_y \\ \hat{r}_z \end{bmatrix}.$$

(Check this for yourself: Write out the dot product on the left in terms of the Cartesian components, and do the matrix multiplication on the right. These should agree). Note that the matrix in the middle depends only on the coordinates of the charges (the primed variables), while the vectors on the ends depend only on the coordinates observation.

Finally we are in a position to write out the V_2 term in Eq. (6) in a way that is characteristic of the charge distribution,

$$\phi^{(2)}(\mathbf{x}) = \frac{1}{2} \frac{\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}} \cdot \hat{\mathbf{r}}}{r^3}, \tag{9}$$

where $\overleftrightarrow{\mathbf{Q}}^{(2)}$ is the quadrapole *tensor*. It is a 3×3 matrix whose component are

$$Q_{ij} = \int d^3x' (3x'_i x'_j - x'^2\, \delta_{ij})\, \rho(\mathbf{x}'), \text{ where } \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{10}$$

We will return to discuss this is greater detail in the following section.

We have now expressed the potential as

$$\phi(\mathbf{x}) = \underbrace{\frac{q_{net}}{r}}_{\substack{monopole \\ potential}} + \underbrace{\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}}_{\substack{dipole \\ potential}} + \underbrace{\frac{1}{2} \frac{\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}} \cdot \hat{\mathbf{r}}}{r^3}}_{\substack{quadrapole \\ potential}} + \cdots. \tag{11}$$

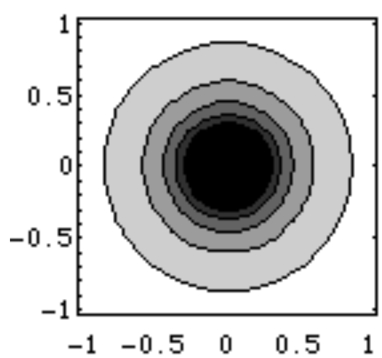
The quantities $\{q_{net}, \mathbf{p}, \overleftrightarrow{\mathbf{Q}}\}$ are characteristics of the *charge distribution*, whose physical meaning we will explore below. Knowing these, we can immediately right down the potential to order $(a/r)^3$, where a is the characteristic size of the charge distribution (see Fig. 1 above). We thus see the charge distribution as decomposing into a pure "monopole field", "dipole field", "quadrapole field", "octapole field", etc. How much of each of these ingredients we need depends on the charge distribution. Of course as we go further away from the distribution $r >> a$, the first **nonvanishing** term dominates (e.g. if q_{net}=0, but the dipole moment is not zero, then far enough away the potential looks like that of a "pure dipole" with the contributions of higher order moments made more and more negligible).

III. Physical Meaning of the multipole moments

As discussed in the introduction, the moments of the distribution tell us something about how the charges are distributed in space. The monopole moment is the easiest to understand;

$$q_{net} = \int d^3x\, \rho(\mathbf{x}) = \sum_n q(n) \tag{13}$$

it's just the total net charge. Here I have dropped the primes inside the integral (we have gotten rid of all reference the position of observation, so there's no need to carry around an extra label). In addition I have include the equivalent expression for an assembly of point charge{q(n)}. For a system with a net charge, then far enough way, the field looks like that of a point charge, with spherically symmetric equipotential surfaces. $\phi^{(0)}(\mathbf{x}) = \dfrac{q_{net}}{r}$



The dipole moment vector represents the average position of the positive charge minus the average position of the negative charge along each coordinate axis.

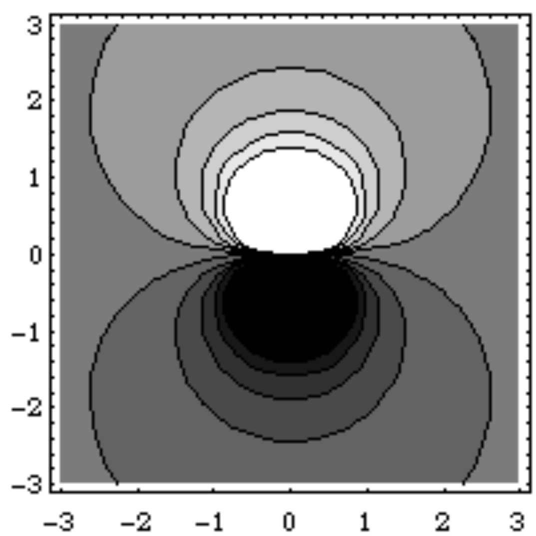
$$
\begin{aligned}
&\mathbf{p} = \int d^3x \;\; \mathbf{x}\rho(\mathbf{x}) = \sum_n \mathbf{x}(n)q(n) \\
&p_x = \int d^3x \;\; x\rho(\mathbf{x}), \;\; p_y = \int d^3x \;\; y\rho(\mathbf{x}), \;\; p_z = \int d^3x \;\; z\rho(\mathbf{x})
\end{aligned}
\quad . \tag{14}
$$

The potential associated with a pure "point dipole" is

$$
\begin{aligned}
\phi^{(1)}(\mathbf{x}) &= \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} \\
&= p_x \frac{x}{r^3} + p_y \frac{y}{r^3} + p_z \frac{z}{r^3} \quad \text{(In Cartesian coordinates)} \\
&= p_x \frac{\sin\theta\cos\phi}{r^2} + p_y \frac{\sin\theta\sin\phi}{r^2} + p_z \frac{\cos\theta}{r^2} \quad \text{(In spherical coordinates)}
\end{aligned}
\tag{15}
$$

There is no such thing as a pure point dipole. However, two equal and opposite point charges separated by a distance s closely approximate the pure dipole at large distances, e.g. a charge +q at $\mathbf{x} = s/2\,\hat{\mathbf{z}}$, and a charge –q at $\mathbf{x} = -s/2\,\hat{\mathbf{z}}$. The dipole moment vector is then $\mathbf{p} = qs\,\hat{\mathbf{z}}$, and the associated potential is:

$$
\phi^{(1)}(\mathbf{x}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = qs\frac{z}{r^3} = qs\frac{\cos\theta}{r^2} \tag{16}
$$



The quadrapole moment is the most complicated object we will deal with. As discussed in the introduction, second order moments give us information about how the distribution is spread out from the origin. The quadrapole moment elements are a special 3D case. We know that if we take a charge q and spread it out uniformly over the volume of a sphere, then outside that sphere, the field just like that of a point charge at the origin. Therefore, after spreading the charge out in a spherically symmetric way we still have only a monopole moment. Physically, the quadrapole moment tells us the degree to which the positive and negative charge distributions are **nonspherical** . Quadrapole moments are important in many fields of physics, including nuclear, atomic, and astrophysics. For example, the deviation of the earth from a perfect sphere give rise to a quadrapole gravitational potential seen by the moon. There is a close relationship between the quadrapole tensor and the moment of inertia tensor familiar in rotational dynamics. Basically, the moment of inertial tensor tells us how the mass is distributed about the principle axes. The quadrapole tensor tells us the difference between the given distribution about the principle axes compare with a spherical distribution. Of course this is complicated by the fact hat there are two "species" of charge (plus and minus), whereas for mass the is only one sign. For a detailed discussion see, e.g., Goldstein's text on Classical Mechanics.

The components of the quadrapole tensor are written are written compactly in Eq. (10). These are elements of a *symmetric, traceless,* matrix. That is

$$Q_{ij}^{(2)} = Q_{ji}^{(2)}, \quad \text{and } \ \mathrm{Trace}(\overleftrightarrow{\mathbf{Q}}^{(2)}) = Q_{xx}^{(2)} + Q_{yy}^{(2)} + Q_{zz}^{(2)} = 0. \tag{18}$$

Therefore, for the most general distribution, this matrix has only 5 independent elements. If the distribution is cylindrically symmetric about *z,* the off-diagonal elements vanish

Given these elements, the quadrapole potential at the observation point **r** is generally very messy (yuk!)

$$\phi^{(2)}(\mathbf{r}) = \frac{1}{2}\left(\frac{\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}\cdot\hat{\mathbf{r}}}{r^3}\right) = \frac{1}{2}\left(\frac{\mathbf{x}\cdot\overleftrightarrow{\mathbf{Q}}\cdot\mathbf{x}}{r^5}\right) =$$

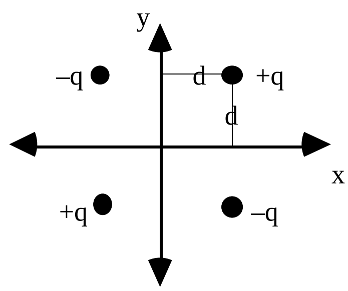
$$= \frac{1}{2}\left(Q_{xx}\frac{x^2}{r^5} + Q_{yy}\frac{y^2}{r^5} + Q_{zz}\frac{z^2}{r^5} + Q_{xy}\frac{2xy}{r^5} + Q_{xz}\frac{2xz}{r^5} + Q_{yz}\frac{2yz}{r^5}\right) \tag{19a}$$

(Cartesian coordinates)

$$= \frac{1}{2}\Bigg(Q_{xx}\frac{\sin^2\theta\cos^2\phi}{r^3} + Q_{yy}\frac{\sin^2\theta\sin^2\phi}{r^3} + Q_{zz}\frac{\cos^2\theta}{r^3} + Q_{xy}\frac{\sin^2\theta\sin 2\phi}{r^3} + Q_{xz}\frac{\sin 2\theta\cos\phi}{r^3} + Q_{yz}\frac{\sin 2\theta\sin\phi}{r^3}\Bigg) \tag{19b}$$

(Spherical Coordinates)

There are no pure point electric quadrapoles in nature. A "physical quadrapole" refers to a charge distribution whose monopole moment and dipole vector moment vanish, e.g. the four charges on the square shown below.



• The net charge is zero --> no monopole moment,

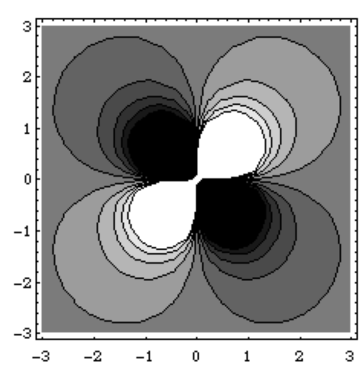
• The (average position of the positive charge) – (average position of the positive charge)=0 ---> No dipole moment.

Neither the positive, nor negative charge distributions are spherically symmetric, so we have a quadrapole moment. The calculation was done in class, with the result

$$\overleftrightarrow{\mathbf{Q}}^{(2)} \doteq \begin{bmatrix} 0 & 12qd^2 & 0 \\ 12qd^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, $Q_{xy} = Q_{yx} = 12qd^2$, with all other moments vanishing. The quadrapole potential associated with this distribution is $\phi^{(2)}(\mathbf{r}) = \frac{1}{2} Q_{xy} \frac{2xy}{r^5} = 12qd^2 \frac{xy}{r^5} = 12qd^2 \frac{\sin^2\theta \sin 2\phi}{r^3}$.

A plot in the x-y plane (z=0, or θ=π/2) is shown below



(see also, the handout "Multipole Moments - Plots").

IV. Doing Calculations

If you have a collection of point charges, label each with one with some index *n*, and there associated position vector. Remember the physical meaning of each moment. The monopole moment is easy. The dipole moment will vanish if: (the weighted average position of the positive charges)-(the weighted average position of the negative charges)=0. Calculating the quadrapole moments requires being organized. Once you have the moments, **forget about the positions of the charges**. The nontrivial potentials are given in Eqs. (15) and (19). For details on a specific examples see Problem Set #3.

When you have a continuous distribution of charges you have to do some integrals. Think about the proper coordinate system to use. For the special case that the distribution has *azimuthal symmetry* about the z-axis, we showed in Problem Set #3, Problem 4,

$$p_x = p_y = 0, \;\; Q_{xy} = Q_{xz} = Q_{yz} = 0, \text{ and } Q_{xx} = Q_{yy}.$$

This make geometric sense. If the distribution is symmetric about the z-axis, the distribution cannot have a dipole moment along x or y (how could we pick a direction in the x-y plane if all are equivalent?) The same holds true for the off diagonal elements of the quadrapole matrix. In this case we find the potential has the form,

$$\phi(r,\theta) = q_{net}\frac{1}{r} + p_z\frac{\cos\theta}{r^2} + Q_{zz}\frac{P_2(\cos\theta)}{r^3} + \ldots.$$

For this special case, all we need to calculate is one number for each multipole. This is of course the spherical coordinate solution to Laplace's equation with azimuthal symmetry. If we obtained the potential in this form through some known boundary conditions, we could just **read-off** the multipole moments directly.

For details on a specific example see Problem Set #3, Problem 4.