

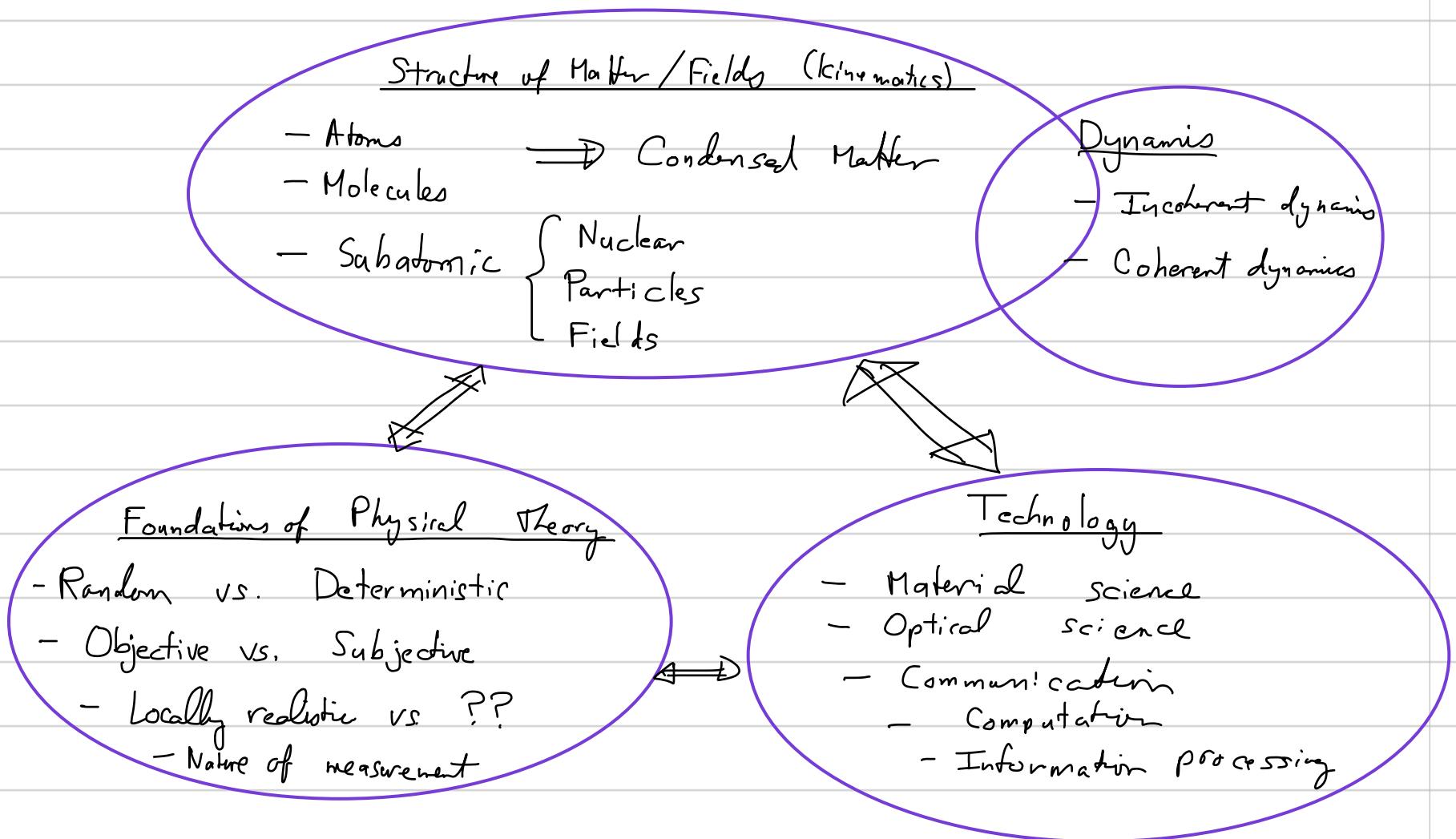
Physics 521: Quantum Mechanics I

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What is Quantum Mechanics About?

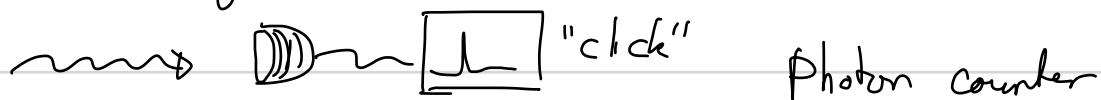
Quantum Physics is the heart of modern physics, but after more than 100 years after Bohr's introduction of the atom model in 1913, the domain and reach of the quantum continues to grow.



The Key Idea: Probability Amplitude

Events and Processes:

An event is registered as a "click" on a detector:



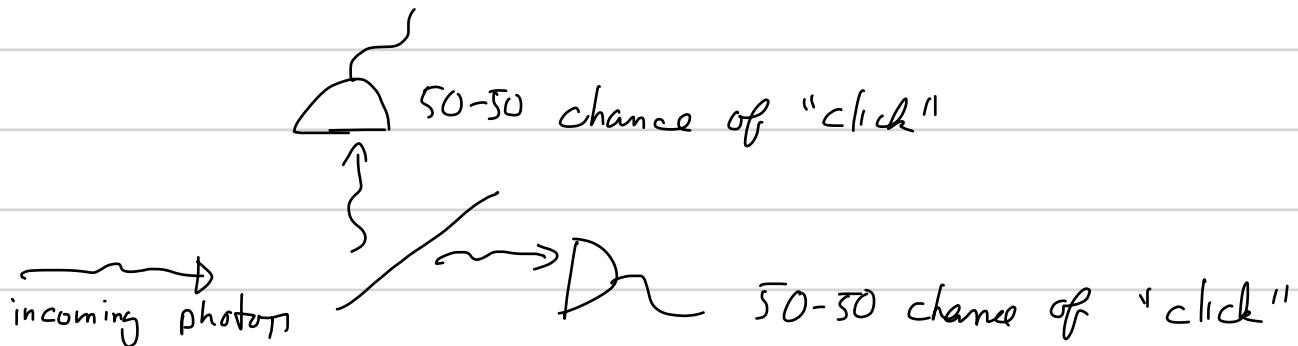
This is a measured event.

Generally, we cannot predict if and/or when an event will happen. The event is "random"

(2)

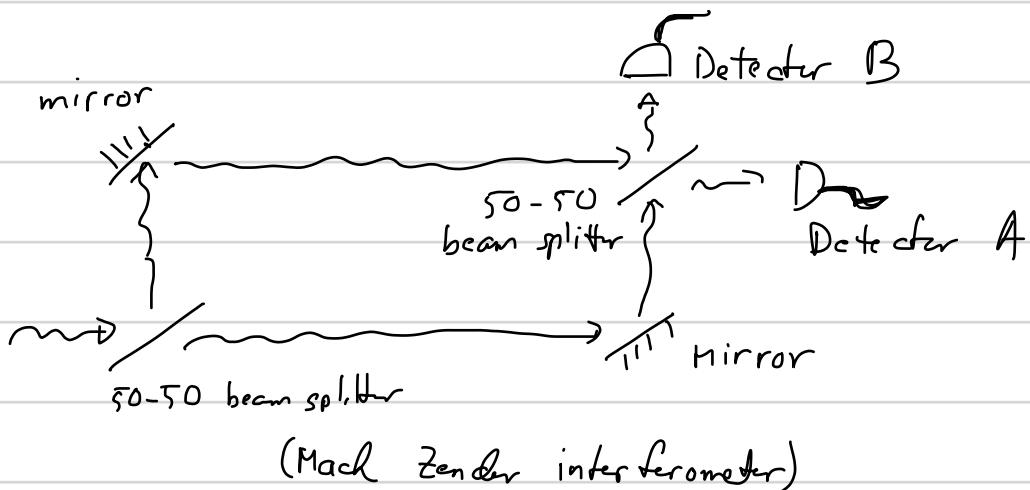
Because we can't predict the event with certainty, we must assign probability to the occurrence of the event.

Example: 50-50 beam splitter



This type randomness looks like flipping a coin. But is it? In a coin-flip the outcome is random due to missing information. We do not know the exact force or torque with which the coin was flipped nor the air pressure, nor have we studied the forces exerted by the surface on which the coin lands. But where is the missing information about the path of the photon at the beam splitter?

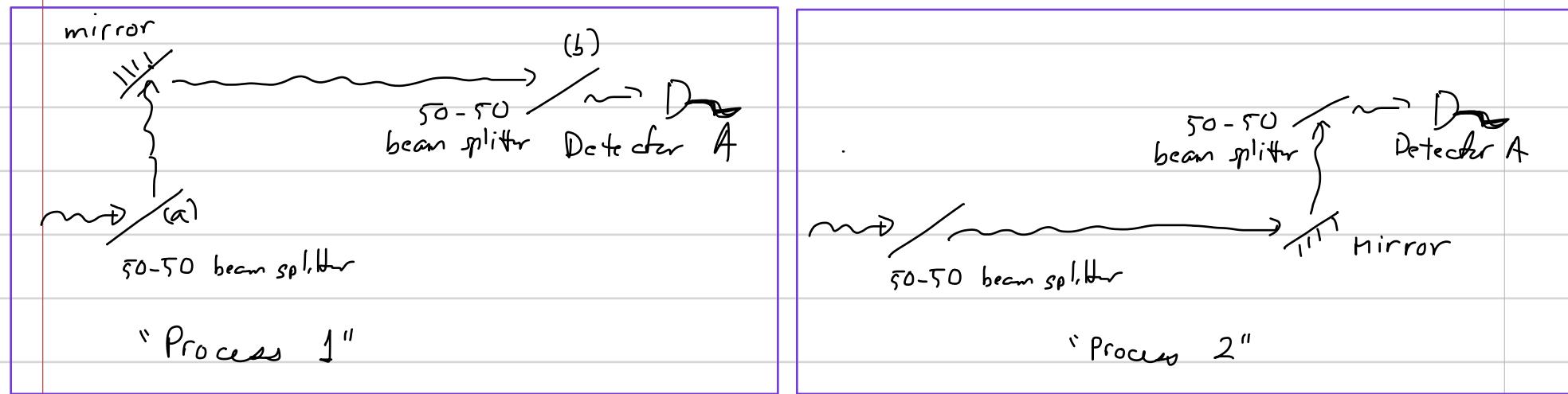
The quantum mechanical answer is that there exist no missing information. The event is fundamentally random. This type of randomness is fundamentally different from the coin flip. For consider the following "interferometer"



If the two arms of the interferometer are equal, then one can show that detector A always clicks with 100% probability and detector B never clicks. This makes no sense logically.

(3)

There are two paths that lead to the event of detector A or B "clicking".



Probability is the language of logic, so to argue logically:

Either Process 1 happened or Process 2

$$\text{Prob A click} = \text{Prob of process 1} + \text{Prob of process 2}$$

Process 1: Photo reflects from B-S (a) and transmits at B-S (b)

$$\text{Prob Process 1} = \text{Prob reflect (a)} \times \text{Prob. trans (b)} = \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\text{Prob Process 2} = \text{Prob. trans (a)} \times \text{Prob. refl. (b)} = \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\Rightarrow \text{Prob A click} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} . \quad \text{But, if we do the experiment, we find the answer is } 1 !$$

The resolution to this paradox is that quantum theory does not operate under our intuitive laws of logic. It has its own "quantum logic." If we have maximum possible knowledge of the state of the system we assign a "probability amplitude," ψ , a complex number with magnitude $|\psi| \leq 1$ and arbitrary phase. The probability of a given event is then given by:

$$\text{Born rule: Probability of event} = |\psi|^2 \quad \text{prob amplitude for event}$$

The counter-intuitive behavior of the Mach-Zehnder interferometer follows from "Feynman's rule"

If two different processes lead to the same final event, but the two processes are indistinguishable, then we must add probabilities amplitudes rather than adding the probabilities of "this" or "that." The total probability of the event is then the square of the total probability amplitude

The two processes 1 and 2 are indistinguishable. By this, we mean that there is no information stored anywhere that tells us whether process 1 or process 2 happened; all we know is that detector A clicked. The total probability amplitude for A clicking is the superposition

$$\Psi_A = \Psi_1 + \Psi_2 \Rightarrow \text{Prob A clicks} = |\Psi_A|^2$$

$$\begin{aligned} \Rightarrow \text{Prob A clicks} &= |\Psi_1 + \Psi_2|^2 = (\Psi_1 + \Psi_2)^* (\Psi_1 + \Psi_2) = |\Psi_1|^2 + |\Psi_2|^2 + \Psi_1^* \Psi_2 + \Psi_2^* \Psi_1 \\ &= \text{Prob of Process 1} + \text{Prob of Process 2} + \text{interference terms} \end{aligned}$$

The cross-terms $\Psi_1^* \Psi_2$ and $\Psi_2^* \Psi_1$ represent quantum interference between different processes.

This interference is the heart of quantum theory, and its mysteries.

$$\Psi_1^* \Psi_2 + \Psi_2^* \Psi_1 = 2 \operatorname{Re} [\Psi_1^* \Psi_2]$$

is a real # (as it must be), but can be positive or negative.

$$\text{In the case of the M-Z interferometer, } 2 \operatorname{Re} [\Psi_1^* \Psi_2] = \frac{1}{2} \Rightarrow \text{Prob A} = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$

This is "constructive" interference. For detector B to click one can show $2 \operatorname{Re} [\Psi_1^* \Psi_2] = -\frac{1}{2}$

$$\Rightarrow \text{Prob B clicks} = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0. \text{ This is "destructive interference."}$$

Classical Interference and Wave-Particle Duality

The phenomenon of interference is familiar from classical physics as a wave phenomenon. Waves scatter at obstacles and can recombine. The total wave amplitude at a given point is the linear superposition of all the waves that impinge on that point. For example, consider an electromagnetic wave with electric field in a linearly polarized plane wave

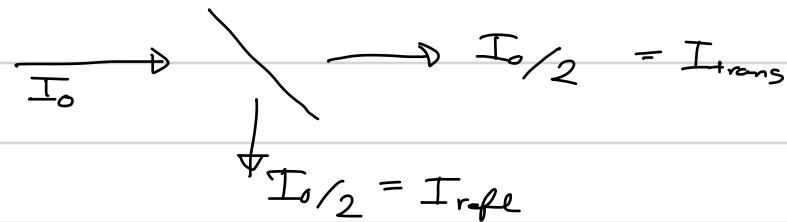
$$\vec{E}(\vec{x}, t) = \vec{E}_0 \cos(kx - \omega t) = \underbrace{\vec{E}_0 e^{i(kx - \omega t)}}_{\text{Complex amplitude}}$$

The intensity of the wave

$$I \sim \langle E^2(x,t) \rangle \sim \bar{E}^*(x,t) \bar{E}(x,t) \text{ like } \psi^* \psi$$

Waves obey the principle of superposition: The total intensity at a given point is not generally the sum of the intensities of all the waves impinging on a given point, but the sum of the wave amplitudes, which then must be squared to obtain the total intensity.

The Born rule is inspired by the relationship between probabilities and frequency of outcomes in a large number of repeated identical experiments. Classically, if we send a laser beam onto a 50-50 beam splitter, half the intensity is transmitted and half reflects



$$\text{Fraction transmitted: } f_{\text{trans}} = \frac{I_0/2}{I_0} = \frac{1}{2} \quad \text{Fraction reflected: } f_{\text{refl}} = \frac{I_0/2}{I_0} = \frac{1}{2}$$

Quantum mechanically we can envision the beam as a stream of photons, which randomly transmit or reflect from the beam splitter with 50-50 chance



With a total of N photons impinging on the beam splitter, the probability of seeing N_A on a given detection is given by the binomial distribution

$$P_M = \binom{N}{N_A} \left(\frac{1}{2}\right)^{N_A} \left(\frac{1}{2}\right)^{N-N_A}$$

On average, the # detected at A: $\langle N_A \rangle = N \frac{1}{2}$. While classically, half the intensity always divides at beam splitter, quantum mechanically, because of the corpuscular nature of photon there are "quantum fluctuations". The variance of number detected @ A

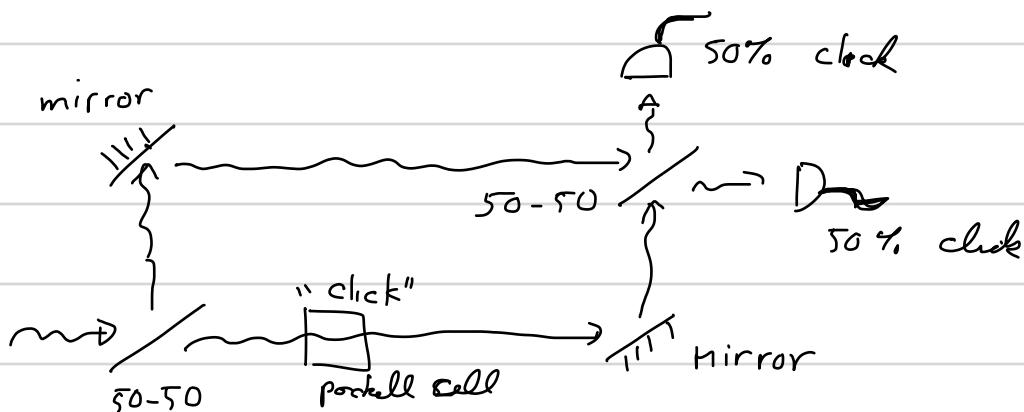
$$\Delta N_A = \sqrt{N} \frac{1}{2}$$

In the limit $N \rightarrow \infty$, all the probability is concentrated on the frequency of outcomes:

$$\text{as } N \rightarrow \infty \quad \frac{\Delta N_A}{\langle N_A \rangle} \rightarrow 0 \Rightarrow \text{fraction of outcomes } f_A = \frac{N_A}{N} \rightarrow \langle N_A \rangle = \frac{1}{2}$$

$$\Rightarrow \frac{\sum_{\alpha}^* E_{\alpha}}{|\Sigma_0|^2} \rightarrow \text{Probability of event.}$$

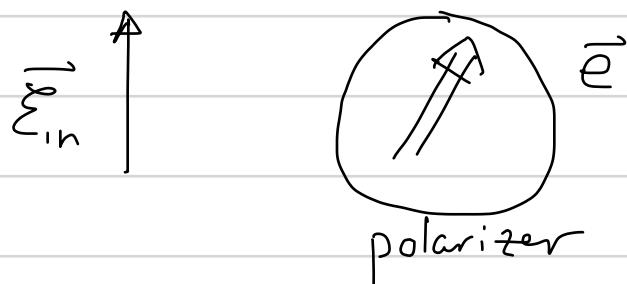
So, are we back to Newton's corpuscular theory of light, with a bit of randomness thrown in? No! Photons do not obey classical trajectories, but are described by probability amplitudes. The photon trajectory in a Mach-Zender interferometer exhibits quantum interference as classical waves. But if we try to "watch" which path the photon takes, and thus make the two photon distinguishable we wipe out this interference



This Bohr referred to a "complementarity." We have two complementary natures: wave and particle. If we look for the particle nature we lose the wave interference. If we see the wave interference, the photon trajectory is not particle-like

Measurement and Projection

There are many properties of light that we might choose to measure. For example, we can measure the polarization of light



$$\vec{e} \cdot \vec{E}_{in} = |\vec{E}_{out}|^2 = \vec{e} \cdot \vec{E}_{in} \cos \theta$$

$$I_{out} = |\vec{E}_{out}|^2 = \cos^2 I_{in}$$

The fraction of the intensity passing through = $\cos^2 \theta$
 Quantitatively, each photon either passes through, or is absorbed by the polarizer. The probability of transmitting = probability of polarization along \vec{e}
 \Leftrightarrow fraction of intensity transmitted = $\cos^2 \theta$

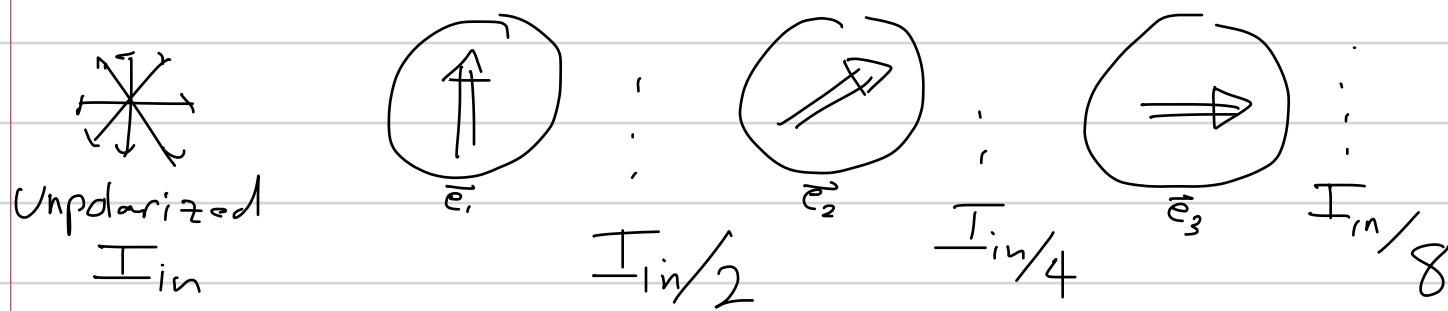
This is known as a **projective measurement** since the polarized state of the photon is "projected" onto the direction of the polarizer. Projective measurements play an important role in the basic building blocks of quantum mechanics, as we will see.

In optics we can block a beam using cross polarizers:



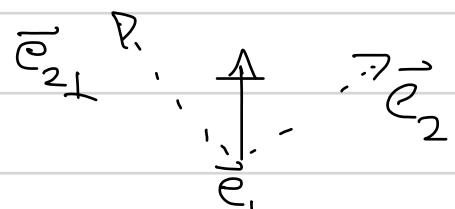
$$I_{\text{out}} = 0$$

But, by putting a polarizer at 45° in between, light again transmits



What does this mean for photons? After the first polarizer we "project" the photon polarization onto the direction \vec{e}_1 ; $\frac{1}{2}$ of the unpolarized photons (on average) pass through \vec{e}_1

The vector \vec{e}_1 is a "superposition" of \vec{e}_2 and $\vec{e}_{2\perp}$



$$\vec{e}_1 = \frac{1}{\sqrt{2}} \vec{e}_2 + \frac{1}{\sqrt{2}} \vec{e}_{2\perp}$$

After the second polarizer, the polarization is \vec{e}_2 . The probability of passing = fraction of intensity passing = $|\vec{e}_2 \cdot \vec{e}_1|^2 = \frac{1}{2}$

Similarly $\vec{e}_2 = \frac{1}{\sqrt{2}} \vec{e}_3 + \frac{1}{\sqrt{2}} \vec{e}_{3\perp}$ \Rightarrow The probability of passing the third polarizer $= |\vec{e}_3 \cdot \vec{e}_2|^2 = \frac{1}{2} \Rightarrow$ Total probability of passing
 $= \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{1}{8}$

Note: Suppose we switch polarizers 1 + 2:



In this case nothing passes. The probability of transmission is zero. These two operations don't "commute" meaning that the order of operations matter. This is a reflection of a kind of "uncertainty principle": The photon's polarization state cannot simultaneously be specified along non-orthogonal directions.