

Physics 521, Quantum I

Lecture 2: Mathematical Foundations: Linear Vector Spaces

We have seen that quantum phenomena involve two fundamental ideas:

- Superpositions of "probability amplitudes"
- Measurement through "projection"

The mathematics that forms the foundation for describing these ideas is linear vector spaces (or more generally "Hilbert space").

To motivate this, let us consider again, the polarization of light. Given a monochromatic E/M plane wave propagating in the z -direction, the most general field is

$$\vec{E}(z,t) = \text{Re}(\underbrace{\vec{E}_0 e^{i(kz - \omega t)}}_{\vec{E}(z,t)})$$

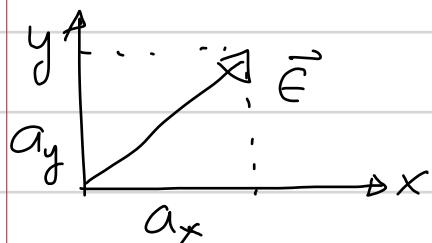
Here, $\vec{E} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y$ is the polarization vector

$\alpha_x = \alpha_x e^{i\phi_x}$, $\alpha_y = \alpha_y e^{i\phi_y}$ are complex numbers

$$\Rightarrow \vec{E}(z,t) = \vec{e}_x (\alpha_x E_0) \cos(kz - \omega t - \phi_x) + \vec{e}_y (\alpha_y E_0) \cos(kz - \omega t - \phi_y)$$

If $\phi_x - \phi_y = m\pi$, $m=0, \pm 1, \pm 2 \dots$ linear polarization, otherwise elliptical.

Consider linear polarization, and take $\phi_x = \phi_y = 0$



$$\vec{E} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y, \quad \alpha_x = \vec{e}_x \cdot \vec{E} \quad (\text{dot or "inner" product})$$

"Normalization" $\|\vec{E}\| \equiv \sqrt{\vec{E} \cdot \vec{E}} = 1$ (unit vector, norm 1)

$$= \alpha_x^2 + \alpha_y^2$$

(2)

If this light is passed through a polarizer along \vec{e}_x , the output field is

$$\vec{\epsilon}_{\text{out}} = \vec{e}_x (\vec{e}_x \cdot \vec{\epsilon}) = \vec{e}_x (\vec{e}_x \cdot \vec{\epsilon}) E_0 e^{ikz - wt}$$

$$I_{\text{out}} = |\vec{\epsilon}_{\text{out}}| = |\vec{e}_x \cdot \vec{\epsilon}|^2 = |\vec{e}_x \cdot \vec{\epsilon}|^2 I_{\text{in}}$$

$$\Rightarrow \text{Fraction of intensity passing beam splitter } f_{\text{out}} = \frac{I_{\text{out}}}{I_{\text{in}}} = |\vec{e}_x \cdot \vec{\epsilon}|^2 = a_x^2$$

At the individual photon level, this translates in the probability that the photon is polarized along x

$$P_x = |\vec{e}_x \cdot \vec{\epsilon}|^2 = a_x^2 \quad (\text{Projective measurement})$$

For example, for linear polarization $\vec{\epsilon} = \frac{1}{\sqrt{2}} \vec{e}_x + \frac{1}{\sqrt{2}} \vec{e}_y$ (45° between x & y) the photon is in a 50-50 superposition of polarization along x and polarization along y $\Rightarrow P_x = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

Generally, for arbitrary elliptical polarization, $\vec{\epsilon}$ is complex

$$I = \vec{\epsilon}^* \cdot \vec{\epsilon} = \vec{\epsilon}^* \cdot \vec{\epsilon} |E_0|^2$$

\Rightarrow We generalized the definition of the inner product and length for a complex vector space $\|\vec{\epsilon}\| = \sqrt{\vec{\epsilon}^* \cdot \vec{\epsilon}}$ ($\vec{\epsilon}^*$ is "dual" vector to $\vec{\epsilon}$)

For example, for circular polarization $\vec{\epsilon}_{\pm} = \frac{1}{\sqrt{2}} \vec{e}_x \pm \frac{i}{\sqrt{2}} \vec{e}_y$

$$\Rightarrow \|\vec{\epsilon}_{\pm}\|^2 = \vec{\epsilon}_{\pm}^* \cdot \vec{\epsilon}_{\pm} = \left(\frac{\vec{e}_x - i \vec{e}_y}{\sqrt{2}} \right) \cdot \left(\frac{\vec{e}_x + i \vec{e}_y}{\sqrt{2}} \right) = \frac{1}{2} (\vec{e}_x \cdot \vec{e}_x + \vec{e}_y \cdot \vec{e}_y) = 1$$

General inner product in a complex vector space:

$$(\vec{u}, \vec{v}) = \vec{u}^* \cdot \vec{v}$$

$$\text{Note: } (\vec{u}, \vec{v})^* = \vec{u} \cdot \vec{v}^* = (\vec{v}, \vec{u})$$

$$\text{Linearity } (\vec{u}, \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 (\vec{u}, \vec{v}_1) + \alpha_2 (\vec{u}, \vec{v}_2)$$

$$(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2, \vec{v}) = \alpha_1^* (\vec{u}_1, \vec{v}) + \alpha_2^* (\vec{u}_2, \vec{v})$$

Example: $(\vec{e}_-, \vec{e}_+) = \vec{e}_-^* \cdot \vec{e}_+ = \frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y) \cdot (\vec{e}_x + i\vec{e}_y) = \frac{1}{2}(\vec{e}_x \cdot \vec{e}_x - \vec{e}_y \cdot \vec{e}_y) = 0$
 $\Rightarrow \vec{e}_+$ and \vec{e}_- are orthogonal

Basis: A set of linearly independent vectors that "span the space"
 \Rightarrow Any vector can be written as a linear superposition of basis vectors.
 Of particular interest are "orthonormal bases" in which the elements of the basis are mutually orthogonal and normalized to have unit length.

E.g. $\{\vec{e}_x, \vec{e}_y\}$ for an orthonormal basis of polarization for z-propagating wave

$$\text{General } \vec{E} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y, \quad \alpha_x = \vec{e}_x \cdot \vec{E}, \quad \alpha_y = \vec{e}_y \cdot \vec{E}$$

$$\Rightarrow \vec{E} = \vec{e}_x \vec{e}_x \cdot \vec{E} + \vec{e}_y \vec{e}_y \cdot \vec{E} = \underbrace{(\vec{e}_x \vec{e}_x + \vec{e}_y \vec{e}_y)}_{\equiv \hat{1}} \cdot \vec{E}$$

Identity tensor $\hat{1} = \vec{e}_x \vec{e}_x + \vec{e}_y \vec{e}_y$ (resolution the identity)
 "Outer product" (dyadic)

$$\vec{e}_x \vec{e}_x = \hat{P}_x \text{ (projection along x-axis).} \quad \hat{P}_x \cdot \vec{E} = \vec{e}_x (\vec{e}_x \cdot \vec{E}) = \alpha_x \vec{e}_x$$

$$\hat{1} = \hat{P}_x + \hat{P}_y \text{ (the identity is resolved by a projection onto all orthogonal directions)}$$

$$\text{Another basis: } \{\vec{e}_+, \vec{e}_-\}, \quad \vec{E} = \alpha_+ \vec{e}_+ + \alpha_- \vec{e}_-, \quad \alpha_+ = \vec{e}_+^* \cdot \vec{E}, \quad \alpha_- = \vec{e}_-^* \cdot \vec{E}$$

$$\Rightarrow \vec{E} = \vec{e}_+ \vec{e}_+^* \cdot \vec{E} + \vec{e}_- \vec{e}_-^* \cdot \vec{E} = \underbrace{(\vec{e}_+^* \vec{e}_+ + \vec{e}_-^* \vec{e}_-)}_{\hat{1}} \cdot \vec{E}$$

(Another resolution of the identity)

The basis vectors vectors are sometimes called a "complete set" because projection onto all directions form a resolution of the identity.

Matrix representation

Given a basis, we can form a "representation" of a vector by arranging the expansion coefficients in a column (1x2 matrix)

$$\vec{e} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y \stackrel{\text{"represented by"}}{\doteq} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} = \alpha_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \stackrel{\doteq}{=} \vec{e}_x \quad \stackrel{\doteq}{=} \vec{e}_y$$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ are the orthonormal basis vectors in represented in that same basis

Inner product: Let $\vec{u} = \alpha \vec{e}_x + \beta \vec{e}_y$, $\vec{v} = \gamma \vec{e}_x + \delta \vec{e}_y$

$$(\vec{u}, \vec{v}) = \alpha^* \gamma + \beta^* \delta = \underbrace{[\alpha^* \beta^*]}_{\text{Dual } \vec{u}^*} \underbrace{[\gamma \delta]}_{\text{Standard rules of matrix multiplication}}$$

Change of basis: Two different representations of the same vector are related

E.g. $\vec{e} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y = \alpha_+ \vec{e}_+ + \alpha_- \vec{e}_-$, how are $\{\vec{e}_x, \vec{e}_y\}$ and $\{\vec{e}_+, \vec{e}_-\}$ related?

Let us represent everything in the $\{\vec{e}_x, \vec{e}_y\}$ basis

$$\vec{e} \stackrel{\doteq}{=} \alpha_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \stackrel{\doteq}{=} \vec{e}_x \quad \stackrel{\doteq}{=} \vec{e}_y = \alpha_+ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} + \alpha_- \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} \stackrel{\doteq}{=} \vec{e}_+ \quad \stackrel{\doteq}{=} \vec{e}_-$$

$$\Rightarrow \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}}_{U: \text{Change of basis matrix}} \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} \quad (\text{Similarity transformation from } \vec{e}_+/\vec{e}_- \text{ to } \vec{e}_x/\vec{e}_y)$$

The columns of \hat{U} are the representation of the old basis vectors in the new basis. Because we change from orthonormal to orthonormal basis, the columns are orthogonal \Rightarrow unitary matrix

A very useful shortcut in performing a change of basis is to use the resolution of the identity:

$$\vec{E} = \hat{1} \cdot \vec{E} = \hat{1} \cdot (\alpha_x \vec{e}_x + \alpha_y \vec{e}_y) = (\underbrace{\vec{e}_+ \vec{e}_+^* + \vec{e}_- \vec{e}_-^*}_{\text{Resolution to change to } \vec{e}_+, \vec{e}_-}) \cdot (\alpha_x \vec{e}_x + \alpha_y \vec{e}_y)$$

$$\Rightarrow \vec{E} = \vec{e}_+ \underbrace{(\vec{e}_+^* \cdot \vec{e}_x \alpha_x + \vec{e}_+^* \cdot \vec{e}_y \alpha_y)}_{\alpha_+} + \vec{e}_- \underbrace{(\vec{e}_-^* \cdot \vec{e}_x \alpha_x + \vec{e}_-^* \cdot \vec{e}_y \alpha_y)}_{\alpha_-}$$

$$\begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{e}_+^* \cdot \vec{e}_x & \vec{e}_-^* \cdot \vec{e}_y \\ \vec{e}_+^* \cdot \vec{e}_y & \vec{e}_-^* \cdot \vec{e}_y \end{bmatrix}}_{\text{Matrix}} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}$$

U for changing $\{\vec{e}_x, \vec{e}_y\}$ to $\{\vec{e}_+, \vec{e}_-\}$

Dirac notation

The writing of vectors, duals, dyadics, etc., can be quite tedious. In the establishment of the founding of modern quantum mechanics, P.A.M. Dirac established a simplifying notation that is universally used (now called "Dirac notation")

- Vector in a complex vector space of arbitrary dimension (Hilbert space \mathcal{H}) "kets" $|v\rangle$.
- Superposition: $|w\rangle = \alpha|u\rangle + \beta|v\rangle$
- Dual vector space: "bras" $\langle u| = |u\rangle^*$ adjoint
 $\langle w| = \alpha^* \langle u| + \beta^* \langle v|$
- Inner product: $(u, v) \equiv \langle u|v\rangle = \langle u|v\rangle$ "braket"
 $\langle u|v\rangle^* = \langle v|u\rangle$, Norm $\|v\| = \sqrt{\langle v|v\rangle}$, Orthogonal $\langle u|v\rangle = 0$

Basis: $\{|e_i\rangle \mid i=1, 2, \dots, d\}$ $d = \text{dimension of Hilbert space}$
 (can be ∞)

Linear operator: $\hat{O} : \mathcal{H} \Rightarrow \mathcal{H}$

$$\hat{O}(\alpha|u\rangle + \beta|v\rangle) = \alpha \hat{O}|u\rangle + \beta \hat{O}|v\rangle$$

Projection operator: $\hat{P}_i = |e_i\rangle\langle e_i|$

Resolution of the identity: Given a "complete set"
 (orthonormal basis $\{|e_i\rangle\}$): $\hat{1} = \sum_{i=1}^d \hat{P}_i = \sum_{i=1}^d |e_i\rangle\langle e_i|$

$$\langle e_i | e_j \rangle = \delta_{ij}$$

Representations: Insert a complete set

$$|v\rangle = \hat{1}|v\rangle = \sum_i |e_i\rangle \underbrace{\langle e_i | v \rangle}_{\alpha_i} = \sum_i \alpha_i |e_i\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix}$$

Change of basis: Suppose $\{|f_i\rangle\}$ is another orthonormal basis

$$\Rightarrow \langle f_i | f_j \rangle = \delta_{ij} \quad \sum_{i=1}^d |f_i\rangle \langle f_i| = \hat{1}$$

$$|v\rangle = \sum_{j=1}^d \underbrace{|f_j\rangle \langle f_j | v \rangle}_{\beta_j} = \sum_j \beta_j |f_j\rangle \stackrel{\text{insert complete set } \hat{1}}{=} \sum_{i,j} \beta_j |e_i\rangle \langle e_i | f_j \rangle$$

$$\Rightarrow |v\rangle = \sum_i \alpha_i |e_i\rangle = \sum_i \left(\sum_j \langle e_i | f_j \rangle \beta_j \right) |e_i\rangle$$

$$\Rightarrow \alpha_i = \sum_j \langle e_i | f_j \rangle \beta_j = \sum_j U_{ij} \beta_j$$

$U_{ij} = \langle e_i | f_j \rangle$: elements of a unitary matrix