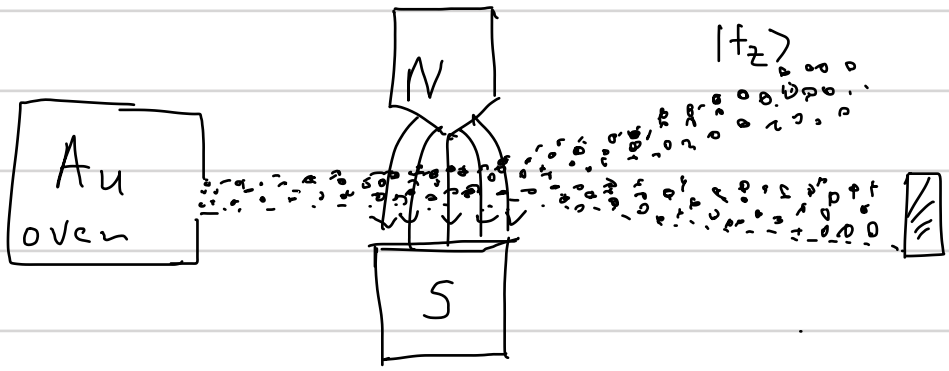
Physics 521, Quantum I

Lecture 7: The Density Operator and Decoherence

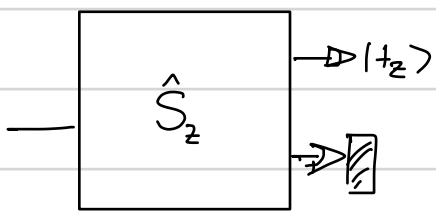
Pure States

We have defined pure states as quantum states in which we have "maximum possible information" about the system. How do we prepare a pure state? We have seen one example last lecture in the context of Stern-Gerlach

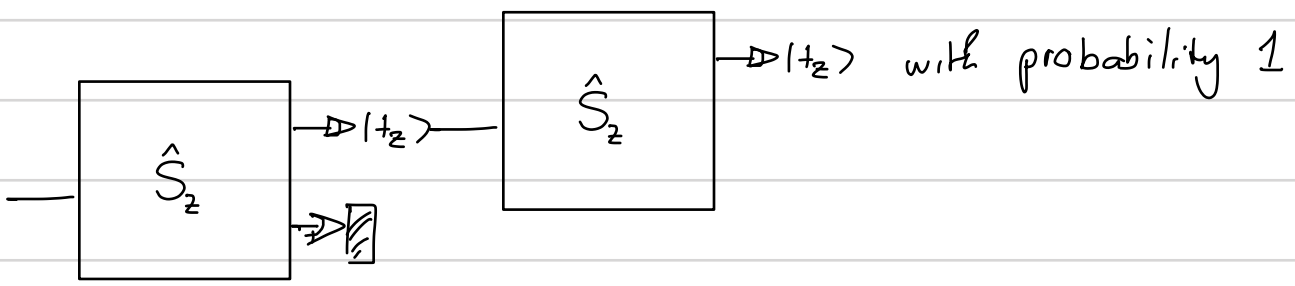


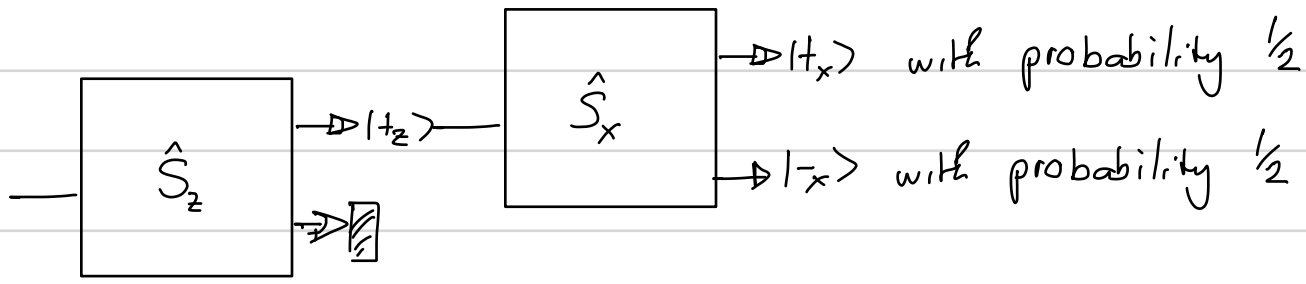
The atoms in the upper path are prepared in the pure state $|\uparrow_z\rangle$, spin-up along z. These are prepared as the outcome of a projective measurement, in a state with a nondegenerate eigenvalue. As such, after the measurement, we are guaranteed that the atom is in the state $|\uparrow_z\rangle$ - It is a fine-grained measurement, with no better possible resolution.

To simply the notation, let as replace the S-G device with a box that denote the observable being measured. (assumed a strong projective measurement).



Though, the state is pure, the not every measurement we do to follow will have a definite outcome. Some will, some won't.

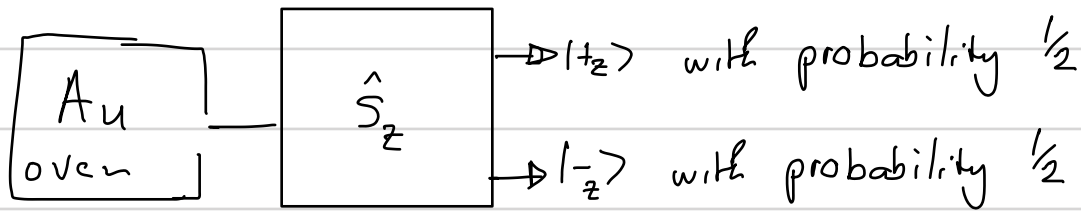


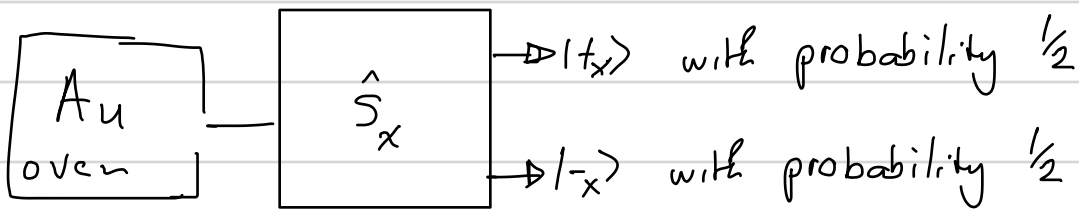


This is the weird thing about quantum systems. We cannot have complete information about all non-compatible variables.

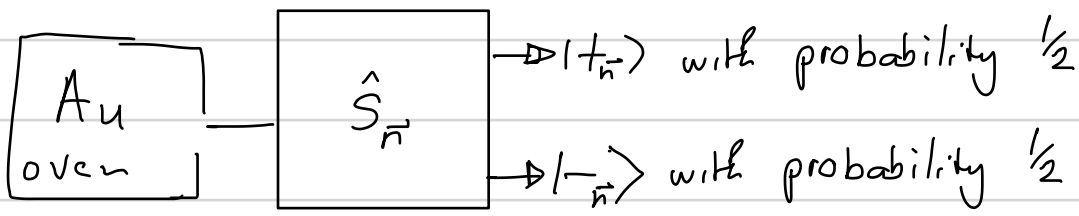
Mixed States

What is the state we assign to the atoms coming out of the oven?
In some sense it is completely random



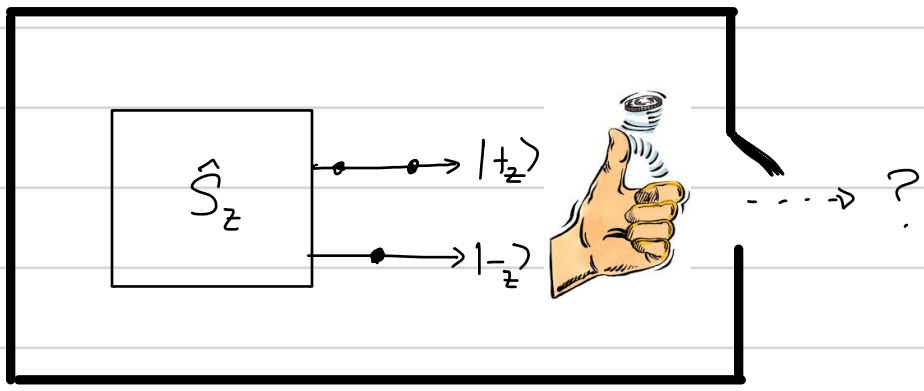


In fact for an arbitrary direction of analysis,



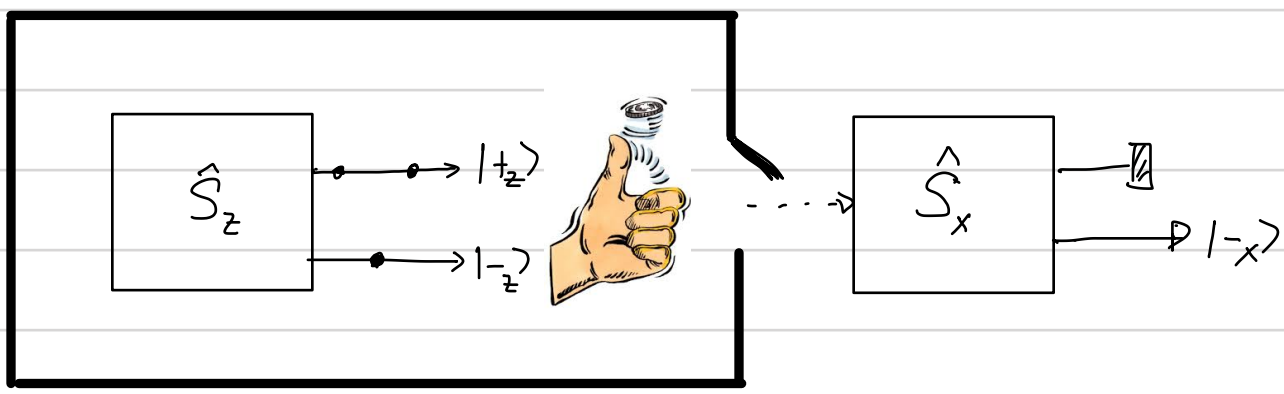
The job of the quantum mechanical state is to provide a mathematical tool with which to predict the outcome of measurements. What kind of state predicts the outcomes above?

Consider another preparation scheme. Suppose the preparer has a Stern–Gerlach device inside a black box that she select either $|+_z\rangle$ or $|-_z\rangle$ spins. She then flips a coin and decides which spin to set to me to then measure.



The preparer tells us the weighting of the coin: With probability P_{+_z} she send $|+_z\rangle$ with probability P_{-_z}, she sends $|-_z\rangle$. The state is not a pure state. It is a statistical mixture of $|+_z\rangle$ and $|-_z\rangle$. We say it is a mixed state. because we do not a complete information about the preparation. That information exists (with the preparer); she's just not telling us.

If we measure $\hat{S}_z$, then we know that we will find $|+_z\rangle$ or $|-_z\rangle$ with probabilities P_{+_z} and P_{-_z} respectively. If this is a fair coin, then these are the same outcomes we would see for the pure state $|-_x\rangle = \frac{1}{\sqrt{2}}|+_z\rangle + \frac{1}{\sqrt{2}}|-_z\rangle$. But the state coming out of this black box is clear not $|+_x\rangle$, for if we measure along the x-axis, we do not get a definite answer.



To calculate the probability of, say, $|-_x\rangle$ we need to consider the weighted sum of the two possible preparations:

$$\text{Probability of finding } -_x = \underbrace{|\langle -_x|+_z\rangle|^2}_{\text{probability of } |-_x\rangle \text{ given } |+_z\rangle} \underbrace{P_{+_z}}_{\text{probability she sent } |+_z\rangle} + \underbrace{|\langle -_x|-_z\rangle|^2}_{\text{probability of } |-_x\rangle \text{ given } |-_z\rangle} \underbrace{P_{-_z}}_{\text{probability she sent } |-_z\rangle}$$

This is a kind of "classical probability theory" corresponding to two alternative outcomes, argued via conditional logic

$$p_{\text{measure}|-x\rangle} = p(-_x|+_z)\,p(+_z) + p(-_x|-_z)\,p(-_z)$$

Here $p(-_x|\pm_z) = |\langle -_x|\pm_z\rangle|^2 = \frac{1}{2}$

$$\Rightarrow p_{\text{measure}|-x\rangle} = \frac{1}{2}\left(p(+_z) + p(-_z)\right) = \frac{1}{2}$$

In contrast, for the pure state $|+_x\rangle = \frac{1}{\sqrt{2}}|+_z\rangle + \frac{1}{\sqrt{2}}|-_z\rangle$

$$p_{\text{measure}|-x\rangle} = |\langle -_x|+_x\rangle|^2 = 0$$

The distinguishing feature between the 50-50 statistical mixture of $|\pm_z\rangle$ and the coherent superposition of $|\pm_z\rangle$ is interference

$$p_{\text{measure}|-x\rangle} = |\langle -_x|\left(\tfrac{1}{\sqrt{2}}|+_z\rangle + \tfrac{1}{\sqrt{2}}|-_z\rangle\right)|^2$$

$$= \tfrac{1}{2}|\langle -_x|+_z\rangle|^2 + \tfrac{1}{2}|\langle -_x|-_z\rangle|^2 \qquad \text{(like classical logic)}$$

$$+ \tfrac{1}{2}\langle -_x|+_z\rangle\langle -_z|-_x\rangle + \tfrac{1}{2}\langle -_x|-_z\rangle\langle +_z|-_x\rangle \qquad \text{(interference terms)}$$

$$= \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}\right)}_{\frac{1}{2}} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{-1}{\sqrt{2}}\right) + \tfrac{1}{2}\left(\tfrac{-1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)}_{-\frac{1}{2}} = 0 \quad !$$

Destructive interference

Statistical mixtures differ from quantum superposition in that the latter can exhibit interference between different alternatives. This has profound implications for the prediction of measurement outcomes. The notion of statistical mixture vs. coherent superposition is familiar in statistical optics: unpolarized light is a statistical

mixture of horizontal + vertical polarization. Linear polarization at 45° is a coherent superposition of horizontal + vertical.

Density Operator

The way we calculated the probability of a measurement outcome for the state coming out of our black box was tedious. Is there one state that describes the statistical mixture, and allows us to calculate all measurement probabilities?

In the example above, the preparer send $|\pm_z\rangle$ with probability $p_{\pm_z}$. Suppose we make a projective measurement along an arbitrary axis, $\vec{n}$

$$P_{\text{measuring } |+_{\vec{n}}\rangle} = |\langle +_{\vec{n}} | +_z \rangle|^2 p_{+_z} + |\langle +_{\vec{n}} | -_z \rangle|^2 p_{-_z}$$

$$= \langle +_{\vec{n}} | +_z \rangle \langle +_z | +_{\vec{n}} \rangle p_{+_z} + \langle +_{\vec{n}} | -_z \rangle \langle -_z | +_{\vec{n}} \rangle p_{-_z}$$

$$= \langle +_{\vec{n}} | \underbrace{\left(p_{+_z} |+_z\rangle\langle +_z| + p_{-_z} |-_z\rangle\langle -_z| \right)}_{} |+_{\vec{n}}\rangle$$

$$\equiv \hat{\rho} : \text{density operator}$$

$\hat{\rho} = p_{+_z} |+_z\rangle\langle +_z| + p_{-_z} |-_z\rangle\langle -_z|$ is the state describing a statistical mixture of spin up and down along z

This is to be contrasted with $|\psi\rangle = C_{+_z} |+_z\rangle + C_{-_z} |-_z\rangle$, which is a coherent superposition of $|\pm_z\rangle$. Both show the same measurement predictions for a measurement along z when $p_{\pm_z} = |C_{\pm_z}|^2$, but different outcomes along other axis due to interference.

Generally, suppose the preparer sends us one an ensemble of possible states $\{|\psi_i\rangle\}$ with respective probability p_i (roll of the die). The density operator describing the statistical mixture of these state is

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

In a pure state, the preparer sends a definite $|\psi\rangle$

> Pure state $\hat{\rho} = |\psi\rangle\langle\psi|$

So, let us return to state of a spin coming out of the oven. The state is <u>mixed</u>, a <u>thermal state</u> at temperature T

$$\hat{\rho} = \sum_E \frac{e^{-E/k_B T}}{Z} |E\rangle\langle E| :$$ Statistical mixture of energy eigenstates depending on temperature T

<u>Understating the density operator</u>

Given a general state $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, the probabality to find outcome "a" in a projective measurement of $\hat{A}$ (nondegenerate)

$$p_a = \sum_i p_i |\langle a|\psi_i\rangle|^2 = \sum_i p_i \langle a|\psi_i\rangle\langle\psi_i|a\rangle = \langle a|\sum_i p_i |\psi_i\rangle\langle\psi_i|)|a\rangle$$

$$\Rightarrow \boxed{p_a = \langle a|\hat{\rho}|a\rangle : \text{ The diagonal matrix element of } \hat{\rho} = \text{Probability to find } a}$$

What about the off-diagonal matrix elements?

Consider a pure state of a spin-½ particle: $|\psi\rangle = c_{+z}|+z\rangle + c_{-z}|-z\rangle$.

$$\Rightarrow \hat{\rho} = |\psi\rangle\langle\psi| = (c_{+z}|+z\rangle + c_{-z}|-z\rangle)(c_{+z}^*\langle+z| + c_{-z}^*\langle-z|)$$

$$\hat{\rho} = \underbrace{|c_{+z}|^2 |+z\rangle\langle+z| + |c_{-z}|^2 |-z\rangle\langle-z|}_{\text{diagonal elements}} + \underbrace{c_{+z}c_{-z}^* |+z\rangle\langle-z| + c_{-z}c_{+z}^* |-z\rangle\langle+z|}_{\text{off-diagonal elements}}$$

The off-diagonal matrix elements capture the fact the $|\pm_z\rangle$ alternatives can interfere. This distinguishes the coherent superposition from the statistical mixture.

In a matrix representation in the $\{|+_z\rangle, |-_z\rangle\}$ basis

Statistical mixture of $|\pm_z\rangle$

$$\hat{\rho} \doteq \begin{bmatrix} P_+ & 0 \\ 0 & P_- \end{bmatrix}$$

Coherent superposition of $|\pm_z\rangle$

$$\hat{\rho} \doteq \begin{bmatrix} |C_{+_z}|^2 & C_{+_z} C_{-_z}^* \\ C_{-_z} C_{+_z}^* & |C_{-_z}|^2 \end{bmatrix}$$

- The diagonal matrix elements of $\hat{\rho}$ are the "populations" ≡ probability to find that element
- The off-diagonal matrix elements of $\hat{\rho}$ are the "coherences" ≡ the potential to see interference between the two possibilies

Consider a general pure state in a d-dimensional Hilbert space:

$$|\psi\rangle = \sum_{\alpha=1}^{d} C_\alpha |\alpha\rangle \quad \text{basis vector} \qquad \hat{\rho} = |\psi\rangle\langle\psi|$$

Density matrix elements: $\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}|\beta\rangle = \langle\alpha|\psi\rangle\langle\psi|\beta\rangle$

$$\Rightarrow \boxed{\rho_{\alpha\beta} = C_\alpha C_\beta^*}$$

Diagonal element: $\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}|\alpha\rangle = |C_\alpha|^2$

Off-diagonal element: $\rho_{\substack{\alpha\beta \\ \alpha\neq\beta}} = |C_\alpha||C_\beta| e^{i(\phi_\alpha - \phi_\beta)}$ where $C_\alpha = |C_\alpha| e^{i\phi_\alpha}$

information about relative phase of probability amplitudes ⇒ interference

In a general mixed state, $\hat{\rho} = \sum P_i |\psi_i\rangle\langle\psi_i|$
(statistical mixture of different pure states)

$$\Rightarrow \rho_{\alpha\beta} = \sum_i P_i \langle\alpha|\psi_i\rangle\langle\psi_i|\beta\rangle = \sum_i P_i\, c_\alpha^{(i)} c_\beta^{(i)*}$$

$$= \overline{c_\alpha c_\beta^*} \quad \text{(ensemble average of different pure states)}$$

$$= \overline{|c_\alpha||c_\beta|\, e^{i(\phi_\alpha - \phi_\beta)}}$$

If different members of the ensemble have different relative phases, then the ensemble average can <u>wash out</u> the phase difference
$\Rightarrow$ loss of coherence!

<u>Formal Properties of the Density Operator:</u> $\hat{\rho} = \sum P_i |\psi_i\rangle\langle\psi_i|$ (for some ensemble)

- <u>Hermitian</u> $\boxed{\hat{\rho}^\dagger = \hat{\rho}}$

- <u>Positive</u> $\langle\phi|\hat{\rho}|\phi\rangle = \sum_i P_i |\langle\phi|\psi_i\rangle|^2 \geq 0 \quad \forall |\phi\rangle \Rightarrow \boxed{\hat{\rho} \geq 0}$

- <u>Normalization</u> Given an orthonormal basis, $\{|a\rangle\}$, we normalize $\hat{\rho}$ such that $P_a = \langle a|\hat{\rho}|a\rangle$, $\sum_a P_a = \underbrace{\sum_a \langle a|\hat{\rho}|a\rangle}_{\text{trace of } \hat{\rho}} = 1$

$\Rightarrow \boxed{\hat{\rho} \text{ is normalized} \Leftrightarrow Tr(\hat{\rho}) = 1}$

Note: The most general mixed state is specified by $d^2 - 1$ parameters ($d \times d$ matrix $-$ 1 for normalization). If the state is <u>pure</u> we need only $2d - 1$ parameters.

- <u>Diagonalization</u>

Since $\hat{\rho}$ is Hermitian it is a normal operator, and thus can be diagonalized. Since $\hat{\rho} \geq 0 \Rightarrow$ All its eigenvalues are ≥ 0.

$\Rightarrow$ If $\{\lambda_i\}$ are its eigenvalues and $\{|e_i\rangle\}$ its eigenvectors, $\hat{\rho} = \sum \lambda_i |e_i\rangle\langle e_i|$

$\hat{\rho}$ is a statistical mixture of its eigenvectors, with probabilities given by its eigenvalues

• Pure vs. Mixed State:

If $\hat{\rho}$ is a pure state $\Rightarrow \exists |\psi\rangle$ such that $\hat{\rho} = |\psi\rangle\langle\psi|$

$\Rightarrow \hat{\rho}$ has one eigenvector, $|\psi\rangle$, with eigenvalue 1; all other eigenvalues are 0.

$$\hat{\rho}_{\text{Pure}} \doteq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & & \ddots & 0 \end{bmatrix}$$

In the basis of its eigenvectors

For mixed states, in the diagonal representation, $\hat{\rho} = \sum_i \lambda_i |e_i\rangle\langle e_i|$, where $\lambda_i < 1 \ \forall i$

$$\Rightarrow \hat{\rho}^2 = \sum_i \lambda_i^2 |e_i\rangle\langle e_i| \Rightarrow \text{Tr}(\hat{\rho}^2) = \sum_i \lambda_i^2 \equiv \text{"Purity"}$$

Purity:

$\text{Tr}(\hat{\rho}^2) < 1$ for mixed states

$\text{Tr}(\hat{\rho}^2) = 1$ for pure states

"Maximally mixed States" $\Rightarrow$ Equal mixture of all eigenvectors: $\lambda_i = \frac{1}{d} \ \forall i$

$$\Rightarrow \hat{\rho}_{\substack{\text{max} \\ \text{mixed}}} = \sum_{i=1}^{d} \frac{1}{d} |e_i\rangle\langle e_i| = \frac{1}{d} \sum_{i=1}^{d} |e_i\rangle\langle e_i| = \frac{1}{d} \hat{\mathbb{1}}$$

$$\Rightarrow \text{Tr}\left(\hat{\rho}^2_{\substack{\text{max} \\ \text{mixed}}}\right) = \text{Tr}\left(\sum_i \frac{1}{d^2} |e_i\rangle\langle e_i|\right) = \frac{1}{d}$$

$$\underbrace{\frac{1}{d}}_{\text{maximally mixed}} \leq \text{Tr}(\hat{\rho}^2) \leq \underbrace{1}_{\text{pure}}$$

Diagonal vs. off-diagonal matrix elements

We have seen that the diagonal matrix elements of $\hat{\rho}$ are the probabilities to find that vector in a projective measurement, the off-diagonal matrix element are the "coherences" measuring the potential for the two alternatives to interfere. It does make sense to talk about THE off-diagonal elements — this is a basis dependent concept. For example, consider the pure state

$\hat{\rho} = |+_x\rangle\langle+_x|$. In the basis $\{|+_z\rangle, |-_z\rangle\}$

$$\hat{\rho} \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The large off-diagonal elements $\Rightarrow$ $|+_z\rangle$ and $|-_z\rangle$ possibilities are in coherent superposition

In contrast, in the basis $\{|+_x\rangle, |-_x\rangle\}$

$$\hat{\rho} \doteq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

No off-diagonal elements $\Rightarrow$ No superposition of $|\pm_x\rangle$

The off-diagonal elements, thus depend on basis. The exception is for the maximally mixed state.

$\hat{\rho} = \frac{1}{d}\hat{\mathbb{1}} \Rightarrow$ Off-diagonal matrix elements are zero in all bases. A maximally mixed state shows no interference between orthogonal alternatives.

Expectation Values

We showed in homework, $\mathrm{Tr}(\hat{M}|\phi\rangle\langle\psi|) = \langle\psi|\hat{M}|\phi\rangle$

$\Rightarrow \langle a|\hat{\rho}|a\rangle = \mathrm{Tr}(\hat{\rho}\,|a\rangle\langle a|) = \mathrm{Tr}(\hat{\rho}\,\hat{\Pi}_a)$, where $\hat{\Pi}_a = |a\rangle\langle a|$ (1D projector)

This is a form of the Born rule. We can generalize this for degenerate eigenvalues

$$\hat{\Pi}_a = \sum_{i=1}^{g_a} |u_a^{(i)}\rangle\langle u_a^{(i)}| \Rightarrow p_a = \sum_i \langle u_a^{(i)}|\hat{\rho}|u_a^{(i)}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{\Pi}_a)$$

Consider a Hermitian observable with eigendecomposition $\hat{A} = \sum_a a\,\hat{\Pi}_a$

$\Rightarrow$ The expectation value $\langle\hat{A}\rangle = \sum_a a\,p_a = \sum_a a\,\mathrm{Tr}(\hat{\rho}\hat{\Pi}_a) = \mathrm{Tr}\Big(\hat{\rho}\,\underbrace{\sum_a a\,\hat{\Pi}_a}_{\hat{A}}\Big)$

$$\Rightarrow \boxed{\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{A})}$$

General form of the Born rule:

Projective Measurement: $p_a = \langle\hat{\Pi}_a\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{\Pi}_a)$

General POVM: $p_u = \langle\hat{E}_u\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{E}_u)$

Time evolution of the state

In a closed quantum system, the dynamics is described by unitary evolution. Such evolution maps pure states to pure states: $|\psi(t)\rangle = U(t;t_0)\,|\psi(t_0)\rangle \qquad (t > t_0)$.

Written in terms of the density operator

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \hat{U}(t,t_0)\,|\psi(t_0)\rangle\langle\psi(t_0)|\,\hat{U}^{\dagger}(t,t_0) = \hat{U}(t,t_0)\,\hat{\rho}(t_0)\,\hat{U}^{\dagger}(t,t_0)$$

If $\hat{\rho}(t_0)$ is a statistical mixture of pure states at t_0 $\quad \hat{\rho}(t_0) = \sum_i P_i\,|\psi_i(t_0)\rangle\langle\psi_i(t_0)|$

$\Rightarrow$ For closed system evolution each pure state evolves unitarily $\quad |\psi_i(t)\rangle = \hat{U}(t,t_0)\,|\psi_i(t_0)\rangle$

$$\Rightarrow \hat{\rho}(t) = \sum_i P_i\,|\psi_i(t)\rangle\langle\psi_i(t)| = \sum_i P_i\,\hat{U}(t,t_0)\,|\psi(t_0)\rangle\langle\psi(t_0)|\,\hat{U}^{\dagger}(t,t_0) = \hat{U}(t,t_0)\,\hat{\rho}(t_0)\,\hat{U}^{\dagger}(t,t_0)$$

$\Rightarrow$ Regardless of the nature of the state (pure or mixed), if the system is closed, the state evolves according to the unitary transformation

$$\hat{\rho}(t) = \hat{U}(t,t_0)\,\hat{\rho}(t_0)\,\hat{U}^{\dagger}(t,t_0)$$

Unitary evolution preserves the purity of the state

Proof: $\quad \text{Purity}(t) = \text{Tr}\left(\hat{\rho}^2(t)\right) = \text{Tr}\left[\hat{U}(t,t_0)\,\hat{\rho}(t_0)\,\underbrace{\hat{U}^{\dagger}(t,t_0)\,\hat{U}(t,t_0)}_{\hat{\mathbb{1}}}\,\hat{\rho}(t_0)\,\hat{U}^{\dagger}(t,t_0)\right]$

$$= \text{Tr}\left[\hat{U}(t,t_0)\,\hat{\rho}^2(t_0)\,\hat{U}^{\dagger}(t,t_0)\right] = \text{Tr}\left[\hat{U}^{\dagger}(t,t_0)\,\hat{U}(t,t_0)\,\hat{\rho}^2(t_0)\right] = \text{Tr}\left[\hat{\rho}^2(t_0)\right] = \text{Purity}(t_0) \quad \text{q.e.d.}$$

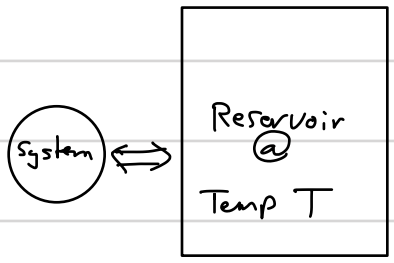
(cyclic property of trace; $\hat{U}^{\dagger}\hat{U} = \hat{\mathbb{1}}$)

In a closed quantum system, information is neither gained nor lost. The purity (mixedness) is a measure of how much information we have about the preparation of the state. If the system is closed, the "state of knowledge" is unchanged.

Decoherence (the loss of coherence)

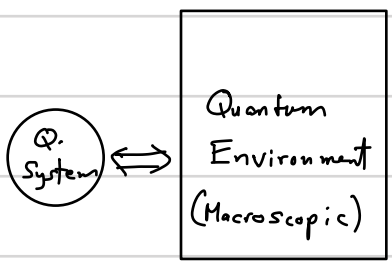
So far we have taken pains to clearly specify that the dynamics we are considering is "closed" system. What we mean by that is that all degrees of freedom are included in the total system. No degrees of freedom are "external" – outside the description of our system. This should be contrasted with a open quantum system in which the system at hand interacts with some degrees of freedom that we don't (or can't) keep track of. These other degrees of freedom often constitute the "environment" of a "thermal reservoir" which is assumed to be macroscopically large and complicated.

Open-systems dynamics is familiar in classical physics. Consider a system in contact with a "bath" or "reservoir" at temperature T in thermal equilibrium.



The system and reservoir can exchange, e.g., energy. The system will eventually, itself, come into thermal equilibrium with the bath. It will have finite entropy (at time T), meaning the system has some randomness (we do not have complete information about our classical system). There are thermal fluctuations. Note, this kind of evolution is typically irreversible; once the system comes into equilibrium it does not spontaneously reassemble from the information lost to the reservoir. The reservoir is so large and complex, that the information is, for all intents and purposes, lost.

Similar scenarios apply in the quantum world. A quantum system is almost always in contact with an "enviroment!" Even in the absence of other matter, a quantum system is it contact with the quantum mechanical vacuum! This too can constitute an "environment" or reservoir.



As in classical mechanics, the quantum system can come into thermal equilibrium with the enviroment. But in the classical world there's something much more. The macroscopic environment can be thought of as performing a measurement on the system but the measure result is hidden. That is degrees of freedom of the environment contain information about the state of system, but we don't have access to these degrees of freedom. Suppose we start with a pure state of the system. After the interaction with the environment, a measurement was done, but we don't know the result! We are missing information. The state becomes mixed

$$\underbrace{|\psi(0)\rangle\langle\psi(0)|}_{\text{pure}} \Longrightarrow \underbrace{\hat{\rho}(t)}_{\text{mixed}}$$

This type of evolution is generally irreverisble for a macroscopic enviroment.

The process whereby a coherent superposition evolves into a statistical mixture is known a decoherence - the loss of quantum coherence. Decoherence follows as an "effective dynamics" of the joint unitary evolution of system + environment, when we loose information to the enviroment. It is not "magic" in the same way as the projection postulate is, anymore than the approach to thermal equilibrium is from the point of view of reversible classical Newtonian laws of motion.

Decoherence explains how a coherent superposition goes to a statistical mixture

$$\hat{\rho}_{pure} = \begin{bmatrix} |c_+|^2 & c_+ c_-^* \\ c_- c_+^* & |c_+|^2 \end{bmatrix} \Longrightarrow \hat{\rho}_{mixed} = \begin{bmatrix} |c_+|^2 & 0 \\ 0 & |c_+|^2 \end{bmatrix}$$

Decoherence can rapidly damp the off-diagonal coherences. At that point we can say the state is effectively $|+\rangle$ or $|-\rangle$, we just don't know which one.

Open quantum systems dynamics are not unitary! We have argued that in its interaction with the environment, a quantum system can lose information and become more mixed. The purity is not conserved $\Rightarrow$ the dynamics are not unitary. Note: through the coupling of a system to a reservoir, we can also increase purity. This is familiar is classical physics, we call it cooling. We can decrease the entropy of a small system by refridgeration. We can get very close (though never exactly) to absolute zero. In the quantum world this is how we obtain some thing close to a pure state — through the techniques of low temperature physics

$$\hat{\rho}_{mix} \longrightarrow |\psi\rangle\langle\psi| \quad \text{(cooling)}$$

How do we describe such non-unitary dynamics? One example is as follows. The environment performs some unitary evolution on the system, but doesn't tell us which one

$$\hat{\rho}(t) = \sum_\mu \underbrace{P_\mu}_{\text{probability that the } \mu^{th} \text{ unitary was performed.}} \hat{U}_\mu(t,t_0)\,|\psi(t_0)\rangle\langle\psi(t_0)|\,\hat{U}_\mu^\dagger(t,t_0)$$
$$= \sum_\mu \hat{A}_\mu(t,t_0)\,|\psi(t_0)\rangle\langle\psi(t_0)|\,\hat{A}_\mu^\dagger(t,t_0)$$

Where $\hat{A}_\mu(t,t_0) = \sqrt{P_\mu}\,\hat{U}_\mu(t,t_0)$ are know as "Kraus operators"

More generally $\hat{A}_\mu = \hat{U}_\mu \sqrt{\hat{E}_\mu}$ (where $\hat{E}_\mu = \hat{A}_\mu^\dagger \hat{A}_\mu$) — polar decomposition

The trace is preserved $Tr(\hat{\rho}(t)) = \sum_\mu \langle\psi(t_0)|\,\hat{A}_\mu^\dagger \hat{A}_\mu\,|\psi(t_0)\rangle = \langle\psi(t_0)|\sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu\,|\psi(t_0)\rangle$

$$= \langle\psi(t_0)|\psi(t_0)\rangle \Rightarrow \sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{\mathbb{1}}$$

The most general dynamical evolution in quantum mechanics is of this form. It is known as a completely positive map

CP-map $\hat{\rho}(t) = \sum_\mu \hat{A}_\mu(t,t_0)\,\hat{\rho}(t_0)\,\hat{A}_\mu^\dagger(t,t_0)$, trace preserving $\sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{\mathbb{1}}$

Special case: Closed system: Single Kraus-operator $\hat{A}_\mu(t,t_0) = \hat{U}(t,t_0)$ (unitary)

Decoherence and the "measurement problem"

Does decoherence — the dynamical process whereby we lose coherence in a quantum system as a we must evolve our state of the system from a pure quantum superposition to a statistical mixture — resolve the mysteries of quantum measurement? This is debateable. In some sense, yes. It is one way to answer the question of the quantum/classical divide. We saw in the Copenhagen interpretation, a measurement happens when a "classical meter" is in contact with the quantum system. But which physical meters should be considered "classical?"

One quantitive way to answer this question is to say that the meter is classical when it is sufficiently macroscopic that the quantum coherence between different "pointer settings" is lost to decoherence. Then we have a statistical mixture of "pointer states" on the meter; They don't not interfere, and can be considered "classical." For all intents and purposes, the "pointer" of the meter is in some state, we just don't know which one — a measurement was "recorded," irreversibly. This is clearly a physical process that happens, as is described within the context of many-body quantum physics. But does it solve the "measurement problem?" For while decoherence can explain how we update our state of knowledge from a pure state to a mixed state, it does not explain how only one outcome randomly occurs. We cannot get randomness out of nothing — we must put it in "by hand." In classical physics, randomness stems from missing information which "exists," we just don't have it. In quantum physics the randomness comes from nowhere. At least in can't be a "hidden variable" local to the system itself. This remains a mystery.