Physics 521
University of New Mexico
I. H. Deutsch

Special Lecture Notes:
Coherent States of a Simple Harmonic Oscillator

The Simple Harmonic Oscillator

Recall the properties of the SHO discussed in lecture. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(\frac{X^2 + P^2}{2}\right) = \hbar\omega\left(\alpha^*\alpha\right),$$

where $X=x/x_c$ and $P=p/p_c$ are dimensionless phase space coordinates with $p_c^2/m = m\omega^2 x_c^2 = \hbar\omega$, and the complex phase space amplitude defined by

$$\alpha \equiv (X + iP)/\sqrt{2}.$$

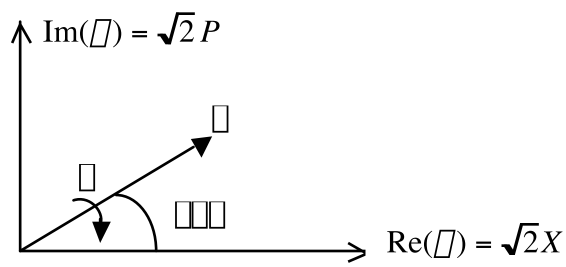
The Hamilton equations of motion have the solutions

$$\begin{aligned}
X(t) &= X(0)\cos\omega t + P(0)\sin\omega t = A\cos(\omega t - \phi) \\
P(t) &= P(0)\cos\omega t - X(0)\sin\omega t = -A\sin(\omega t - \phi) \\
\alpha(t) &= (X(t) + iP(t))/\sqrt{2} = \alpha(0)e^{-i\omega t} = Ae^{i\phi}e^{-i\omega t},
\end{aligned}$$

where the amplitude and phase of the oscillation are determined by the initial conditions

$$\sqrt{2}\,A = \sqrt{\left(X(0)^2 + P(0)^2\right)},\ \ \phi = \tan^{-1}(X(0)/P(0)).$$

These equations of motions are conveniently displayed by a phasor diagram in phase space (the complex α plane)



Thus the magnitude square of the complex amplitude, $\alpha^*\alpha$, is a conserved quantity, whereas the complex amplitude itself is not.

The quantum oscillator follows by the association

$$X \to \hat{X},\ \ P \to \hat{P},\ \ \alpha \to \hat{a} = (\hat{X} + i\hat{P})/\sqrt{2}, \qquad [\hat{X},\hat{P}] = i,\ \ [\hat{a},\hat{a}^\dagger] = 1 .$$

Stationary states of the Hamiltonian are eigenstates of the number operator $\hat{N} \equiv \hat{a}^\dagger \hat{a}$,

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle,\ \ \text{where } \hat{N}|n\rangle = n|n\rangle,\ \ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle,\ \ \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle .$$

The state $|0\rangle$ is the ground state defined by $\hat{a}|0\rangle = 0|0\rangle$. This states has zero average occupation number $\langle n\rangle = \langle 0|\hat{N}|0\rangle = 0$. It is a minimum uncertainty state

$$\Delta X \Delta P = \frac{1}{2}$$

with equal uncertainty in both X and P, $\Delta X = \Delta P = \dfrac{1}{\sqrt{2}}$.

We may now ask, which states of the harmonic oscillator most closely the resemble the classical counterparts? By this we mean that the expectation value of any observable follows the classical trajectory with minimum quantum uncertainty. One quick response might be to apply Bohr's correspondence principle, and consider stationary states with a large occupation number $n \to \infty$. However, in any number state we have

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{P}|n\rangle = 0 .$$

Thus, any stationary state is distinctly nonclassical.

In order to find the quasi-classical states we may take a hint from our quantization procedure. The quantum SHO was defined by associating the classical variables with quantum operators. Thus, a natural choice is to define a state which is an eigenstate of the classical variables. Since it is impossible to find a simultaneous eigenstate of $\hat{X}$ and $\hat{P}$, the best compromise to find "phase-space" eigenstates, that is eigenstates of the complex amplitude operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle .$$

These states are now known as **coherent states** for a historical reason relating to their use in the study of coherent laser fields and quantum optics by Prof. Glauber [See R. Glauber in *Quantum*

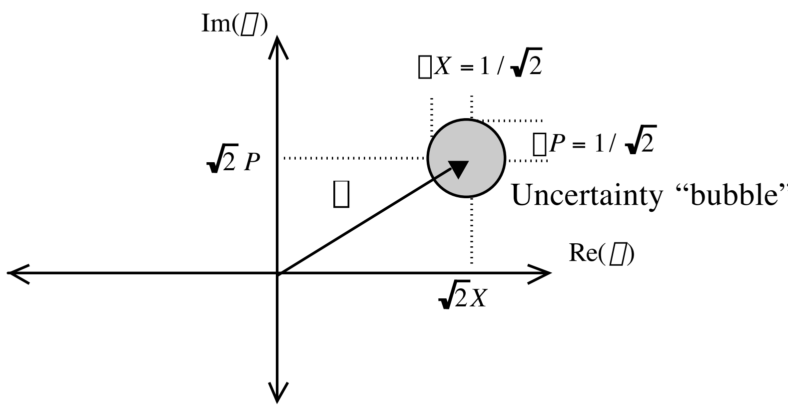
Optics And Electronics, Les Houches Lectures 1964, C. De Witt, A. Bandin, and C. Cohen-Tannoudji editors, Gordon Breach, New York (1965)]. Note: the name coherent state should not be confused with the term "pure state". In this state, expectation values of observables are replaced by their classical values,

$$\langle\alpha|\hat{X}|\alpha\rangle = \langle\alpha|\left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)|\alpha\rangle = \frac{\alpha+\alpha^*}{\sqrt{2}} = X\,,$$
$$\langle\alpha|\hat{P}|\alpha\rangle = \langle\alpha|\left(\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i}\right)|\alpha\rangle = \frac{\alpha-\alpha^*}{\sqrt{2}i} = P\,,$$
$$\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha^*\alpha\,.$$

They are not stationary states of the Hamiltonian since $[\hat{a},\hat{H}] \neq 0$. The coherent states are minimum uncertainty states, with equal uncertainty in X and P,

$$2\Delta X^2 = 2\langle\alpha|\hat{X}^2|\alpha\rangle - 2\langle\alpha|\hat{X}|\alpha\rangle^2 = \langle\alpha|(\hat{a}^\dagger+\hat{a})^2|\alpha\rangle - (\alpha+\alpha^*)^2 = 1$$
$$2\Delta P^2 = 2\langle\alpha|\hat{P}^2|\alpha\rangle - 2\langle\alpha|\hat{P}|\alpha\rangle^2 = -\langle\alpha|(\hat{a}^\dagger-\hat{a})^2|\alpha\rangle + (\alpha-\alpha^*)^2 = 1$$
$$\Rightarrow \quad \Delta X\,\Delta P = \frac{1}{2}$$

The ground state $|0\rangle$ is the only example of a number state which is also a coherent state. The properties of a coherent states are most easily displayed in a phase-space diagram analogous to the classical diagram on page 8.



The mean values are characterized by the solid phasor, and dotted circle represents the quantum uncertainties in the complex amplitude.

Number and phase uncertainties

Although the coherent states are not eigenstates of the number operator, the number states represent a complete basis for the SHO. The coherent states can be expressed as a superposition

$$|\alpha\rangle = \sum_n c_n |n\rangle .$$

Using the eigenstate definition of the coherent state,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \sum_n c_n \hat{a}|n\rangle = \sum_n c_n \sqrt{n}|n-1\rangle .$$

Projecting both sides of the equation with some particular number state $|m\rangle$, we arrive at the recursion relation,

$$\alpha\langle m|\alpha\rangle = \alpha c_m = c_{m+1}\sqrt{m+1} \Rightarrow \quad c_{m+1} = \frac{\alpha}{\sqrt{m+1}} c_m .$$

Thus, $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$. We can determine the constant c_0 by normalization,

$$\langle\alpha|\alpha\rangle = \sum_n |c_n|^2 = \sum_n \frac{|\alpha|^{2n}}{n!}|c_0|^2 = e^{|\alpha|^2}|c_0|^2 = 1$$
$$\Rightarrow |c_0|^2 = e^{-|\alpha|^2} \Rightarrow \quad c_0 = e^{-|\alpha|^2/2} \quad \text{(choose to be real)} .$$

We now have the representation of the coherent state in terms of the number states,

$$|\alpha\rangle = \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ .$$

The probability distribution of occupation number n is given by the absolute square of the expansion coefficients,

$$P_n = |c_n|^2 = e^{-|\alpha|^2} \frac{\left(|\alpha|^2\right)^n}{n!} \ = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!} ,$$

where we have used the fact that $\langle n\rangle = |\alpha|^2$. The probability of occupation number n is thus distributed according to a Poisson distribution. The fluctuation in occupation number is,

$$\Delta N^2 = \langle\alpha|\hat{N}^2|\alpha\rangle - \langle\alpha|\hat{N}|\alpha\rangle^2 = |\alpha|^2 = \langle n\rangle.$$

This result has important implications in quantum optics. A mode of the quantum electromagnetic field is described by a quantum harmonic oscillator. A monochromatic field with a well defined amplitude and phase is described by a "coherent state". The excitation number in that mode represents the number of *photons* in the field. Thus in a coherent state, if we count the number of photons in the mode we will have an uncertainty given by $\sqrt{\langle n\rangle}$. These "counting fluctuations" represent "shot noise" – a purely quantum mechanical effect.

Phase uncertainty of the coherent state

Since these states have an uncertainty in occupation number, we may ask whether there is a canonical conjugate observable to this variable. The natural choice follows from the classical description. On page 8 we defined the complex amplitude in terms of a polar decomposition into amplitude and phase,

$$\alpha = Ae^{i\phi}.$$

Since the number operator is the quantum analog of A^2, the natural choice of the canonically conjugate variable is the phase ϕ. Classically, these are the so called "action-angle" phase-space variables. Quantum mechanically, a brute force quantization in terms of action angle variables in not possible as we shall see.

In the theory of Hilbert spaces it always possible to make a "polar" decomposition of an arbitrary operator, analogous to the polar decomposition of a complex number. In particular, we can decompose the annihilation and creation operators as

$$\hat{a} = \hat{N}^{1/2}\,\hat{e}^{i\phi}, \quad \hat{a}^\dagger = \left(\hat{e}^{i\phi}\right)^\dagger \hat{N}^{1/2}$$

where

$$\hat{N}^{1/2} \equiv \sum_n n^{1/2}|n\rangle\langle n|, \quad \hat{e}^{i\phi} \equiv \sum_n |n\rangle\langle n+1|.$$

Note that the "hat" was placed over the whole operator $\hat{e}^{i\phi}$, rather than ϕ itself. This is because the operator $\hat{e}^{i\phi}$ is not unitary, and thus does not represent the exponentiation of a Hermitian phase operator. It is easy to show that

$$\left[\hat{e}^{i\phi},\left(\hat{e}^{i\phi}\right)^{\dagger}\right]=|0\rangle\langle 0|.$$

Thus, the problem with defining a Hermitian phase operator arises from the fact that there is a lower bound on the occupation number (the ground state). One can however define eigenstates of $e^{i\phi}$,

$$\left|\hat{e}^{i\phi}\right\rangle=\sum_{n=0}^{\infty}e^{in\phi}|n\rangle \Rightarrow \hat{e}^{i\phi}\left|e^{i\phi}\right\rangle=e^{i\phi}\left|e^{i\phi}\right\rangle.$$

The phase probability distribution of an arbitrary state is the given by,

$$P(\phi)=\frac{1}{2\pi}\left|\left\langle e^{i\phi}\middle|\psi\right\rangle\right|^{2}.$$

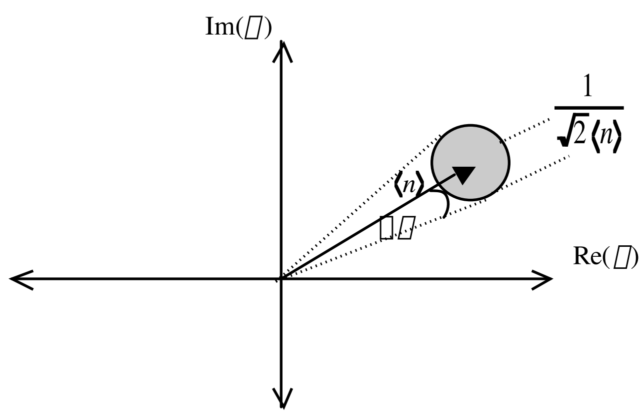
For a number state, $P(\phi)=1/2\pi$, representing a state with a completely uncertain phase. For a coherent state,

$$P(\phi)=\frac{1}{2\pi}\left|\left\langle e^{i\phi}\middle|\alpha\right\rangle\right|^{2}=\frac{1}{2\pi}\left|\sum_{n=0}^{\infty}e^{-in\phi}\langle n|\alpha\rangle\right|^{2}=\frac{e^{-|\alpha|^{2}}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\phi})^{n}}{\sqrt{n!}}\right|^{2}.$$

The terms in the sum are rapidly varying except near $\phi=\mathrm{Arg}(\alpha)$. Thus the average phase will be given by $\mathrm{Arg}(\alpha)$ as expected. In addition one can show that the variance of the phase uncertainty is

$$\Delta\phi^{2}=\frac{1}{4|\alpha|^{2}}=\frac{1}{4\langle n\rangle}.$$

These results make physical sense when viewed from our graphical representation.



Thus in order to create a state with small phase uncertainty requires a large mean excitation number of the coherent state (this is the classical limit). Though in general it is impossible to define a Hermitian phase operator, one can define an approximate one in the limit of large average excitation number since the problem arose from the ground state. In this case we have the approximate uncertainty relation

$$\Delta N^2 \Delta\phi^2 \geq \frac{1}{4}.$$

A coherent state is therefore a minimum number/phase uncertainty state with $\Delta N^2 = \langle n \rangle$, and $\Delta\phi^2 = 1/4\langle n \rangle$.

<u>Phase space displacement operators and the time dependence of coherent states</u>

A very useful way of handling the mathematics associated with coherent states is through the use of the unitary phase space "displacement" operator defined by,

$$D(\alpha) \equiv \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}).$$

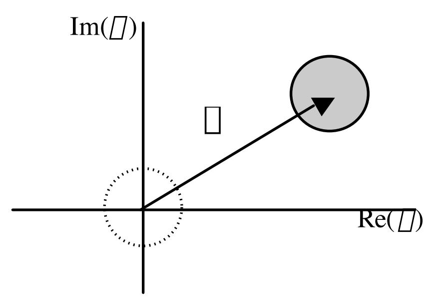
The physical meaning of this operator is clear when we write α in terms of its real and imaginary parts $\alpha = (X + iP)/\sqrt{2}$, $\hat{a} = (\hat{X} + i\hat{P})/\sqrt{2}$

$$D(\alpha) \equiv \exp\left[\frac{(X + iP)(\hat{X} - i\hat{P}) - (X - iP)(\hat{X} + i\hat{P})}{2}\right] = \exp\left\{i\hat{P}X - i\hat{X}P\right\} = \exp\left[\frac{i\hat{p}x - i\hat{x}p}{\hbar}\right].$$

Recall that $\exp\left\{-i\hat{p}x/\hbar\right\}$ is the unitary translation operator in position space, and $\exp\left\{ip\hat{x}/\hbar\right\}$ is the translation operator is momentum space. These operators do not commute. Thus the displacement operator represents a symmetrized translation in phase.

The coherent state is then equal to a unitary transformation on the ground state,

$$|\alpha\rangle = D(\alpha)|0\rangle.$$



To prove this, use the factor that D(α) is a phase-space displacement to show

$$D(\alpha)^{\dagger}\,\hat{a}\,D(\alpha) = \hat{a} + \alpha .$$

Then,

$$D(\alpha)^{\dagger}\,\hat{a}|\alpha\rangle = D(\alpha)^{\dagger}\,\hat{a}\,D(\alpha)|0\rangle = (\hat{a} + \alpha)|0\rangle = \alpha|0\rangle$$
$$\Rightarrow \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

The displacement operator can also be used to easily obtain the decomposition in terms of number states. Using the operator theorem,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}, \text{ when } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0,$$

we get the so call "normal order" decomposition of D(α),

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{\dagger}}e^{-\alpha^{*}\hat{a}}.$$

Then

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{\dagger}}e^{-\alpha^{*}\hat{a}}|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{\dagger}}|0\rangle$$
$$= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n\hat{a}^{\dagger n}}{n!}|0\rangle = \sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

as before.

The unitary operators $D(\alpha)$ form a group known as Heisenberg-Weyl group with the composition law,

$$D(\alpha)D(\beta) = e^{i\,\mathrm{Im}(\alpha\beta^{*})}D(\alpha+\beta).$$

This groups leads to a rich variety of properties of the coherent states and the representation of operators in terms of these states.

Completeness, and representations in terms of coherent states

Given the expansion of the coherent state in terms of number states one can show that these sates form an (over)complete basis,

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1 .$$

We use the term over-complete because these states are not orthogonal

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) + \alpha^*\beta\right] \quad \left|\langle\alpha|\beta\rangle\right|^2 = e^{-|\alpha-\beta|^2} .$$

Although these states are not orthogonal, these become approximately so when $|\alpha-\beta|^2 >> 1$.

Given the completeness of the coherent states, any operator in Hilbert space can be expanded in terms of them. For example, we may consider expansions of the density operator representing a general state of the system,

$$\rho = \int \frac{d^2\alpha}{\pi} P(\alpha,\alpha^*) |\alpha\rangle\langle\alpha| .$$

The c-number function $P(\alpha,\alpha^*)$ is known as the **Glauber-Sudarshan** P-representation. It closely resembles a classical distribution of for an ensemble in phase space. However, this analogy must be treated with some care. In general $P(\alpha,\alpha^*)$ can be negative or highly singular for quantum states with have not classical analog. For a pure coherent state with complex amplitude α_0, the P-representation is, $P(\alpha,\alpha^*) = \pi\delta^{(2)}(\alpha-\alpha_0)$.

The nonclassical states will have singularities worse than a delta function.

Because the coherent states form an over complete basis, the representation of the density operator in terms of them is not unique. An important example is the Wigner function,

$$W(\alpha,\alpha^*) \equiv \int \frac{d^2\beta}{\pi} Tr(\hat{\rho}\hat{D}(\beta))\, e^{\alpha\beta^* - \alpha^*\beta} .$$

In the case of a pure state: $\hat{\rho} = |\psi\rangle\langle\psi|$, with $\alpha = (X + iP)/\sqrt{2}$ this reduces to

$$W(x,p) = \frac{1}{\pi}\int dx'\,\psi^*(x + x')\,\psi(x - x')\,e^{i2px'/\hbar}.$$

Note that $\int W(x, p)\,dp = |\psi(x)|^2$ (probability density in position space)

$\int W(x, p)\,dx = |\tilde{\psi}(p)|^2$ (probability density in momentum space

The Wigner function is always nonsingular, but can be negative, and thus is not generally interpretable as a classical probability distribution on *phase-space* (it is sometimes referred to as a "quasi-probability" distribution). The negative values are an indication of a purely quantum effect - interference of classical paths. In the last few years, techniques have been developed to directly measure the Wigner function for a quantum system. Important examples include the state of a electromagnetic mode [U. Leonhart and H. Paul, Quant. elect. **19**, 89 (1995)], and the state of a single trapped ion whose motion is quantized [C. Monroe et al., Phys. Rev. Lett **75**, 4011 (1995)]. In the latter case the Wigner function for an $|n = 1\rangle$ Fock state was measure, with negative values, indicating nonclassical motion!

The use of the Wigner function (and related functions) has become very prevalent in the last decade or so, because it allows for a clear connection to classical dynamics in phase space. Contemporary examples include: better understanding of the *correspondence principle* (i.e. under what circumstances are the dynamics classical) using *quantum statistical mechanics* (interaction of particles with a "heat bath") [W. H. Zurek, Phys. Today **44**(10), 36 1991], and *quantum chaos* (issues relating to chaos at the quantum level are seen most clearly in a phase space picture in analogy with classical chaotic dynamics) [M. V. Berry, Proc. Roy. Soc. **287**, 30 (1977)].