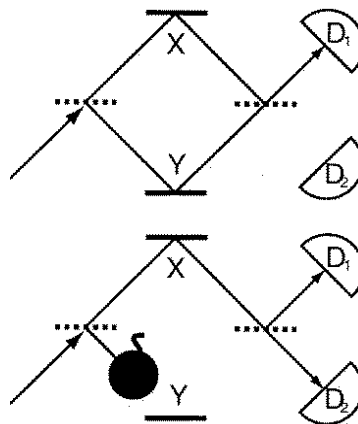# Physics 521: Problem Set #1

Solutions

1. PROBLEM #1

We are asked to detect the presence of a bomb given the interferometric setup shown below.

The upper configuration is just a balanced Mach-Zender interferometer, which was demonstrated in class. To quickly review, the first beamsplitter acts to unitarily transform the states $|u\rangle$, $|d\rangle$, which represent photons coming from above and below the beamsplitter. In particular the input state $|d\rangle$, will be transformed into an equal superposition of photons on the upper (X), and lower (Y) paths. In bra-ket notation its state is then

$$|d\rangle \rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}}\left[|X\rangle + i|Y\rangle\right].$$

Where the i results from reflection of the photon off of the beam splitter. The probability of being in any particular path is

$$|\langle X \text{ or } Y||\Psi\rangle|^2 = \frac{1}{2},$$

as required for a $50 - 50$ beamsplitter. Without the presence of a bomb both photons travel the same distance along path X and Y, and so pick up the same phase, which can be ignored (global phases are ignorible). Then at the final beam splitter the photons are again unitarily transformed, in the same way as before, so

1

that

$$|X\rangle \rightarrow \frac{1}{\sqrt{2}} \left[i\left|D_1\right\rangle + \left|D_2\right\rangle\right],$$

$$|Y\rangle \rightarrow \frac{1}{\sqrt{2}} \left[\left|D_1\right\rangle + i\left|D_2\right\rangle\right],$$

with $|D_i\rangle$ being the path that leads to detector i. Substituting this into the equation for $|\Psi\rangle$ we find that

$$|\Psi\rangle = \frac{1}{2} \left[i\left|D_1\right\rangle + \left|D_2\right\rangle + i\left|D_1\right\rangle - \left|D_2\right\rangle\right] = \left|D_1\right\rangle.$$

Thus without a bomb the photon is always detected by D_1.

When the bomb is present it acts to measure the field in the lower arm of the interferometers Y. In other words, if the bomb explodes then you have no doubt that the photon was in the lower arm (at least for the short amount of time during which such problems still concern you). We can calculate the probability for an explosion (measurement in lower arm) as

$$P(\text{explosion}) = |\langle X||\Psi\rangle|^2 = \frac{1}{2}$$

If there is no explosion then the photon must be on path Y, or equivalently in state $|Y\rangle$. Given this state we can calculate the probability of detection as

$$P(\text{detection by } D_i|Y) = |\langle D_i||Y\rangle|^2 = \frac{1}{2}$$

Which is the same for both detectors. Since the detectors only register when the bomb does not go off, the total probability for detection in detector i will be

$$P(D_i) = P(D_i|\text{no explosion})P(\text{no explosion}) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

Now can we detect a bomb without setting it off. Yes, as we note that $P(D_2|\text{no bomb}) = 0$, while $P(D_2|\text{bomb}) = 1/4$, so that if detector 2 detects, there must be a bomb. Further we see that if there is a bomb detector 2 detects with 25% probability.

2. NIFTY EXTRAS

You may note several shortcomings in the above bomb detection apparatus. Firstly there is a 50% chance that the bomb kills you, which is somewhat undesirable, as detecting the bomb by setting it off is a suboptimal solution. Also, 25% of the time when there is a bomb you receive an inconclusive result(detection by D_1) . This second problem can be overcome by sending through more photons until a conclusive result is reached, unfortunately this results in the bomb exploding 2/3 of the time, and detection only 1/3 of the time. So it seems like we are stuck dying at least 50% of the time.

What we need to do to solve the first problem is consider a beam splitter that is not 50 − 50, such that the arm in which the bomb resides will not have a 50% probability of containing a photon. To do this consider the apparatus depicted on the next page. A single photon is initially input from the lower left, through a shutter. The shutter is then closed and the photon circulates around the cavity M times, at which point the shutter is opened again to let the photon out. At which point it is travelling upwards towards the detection D_1.
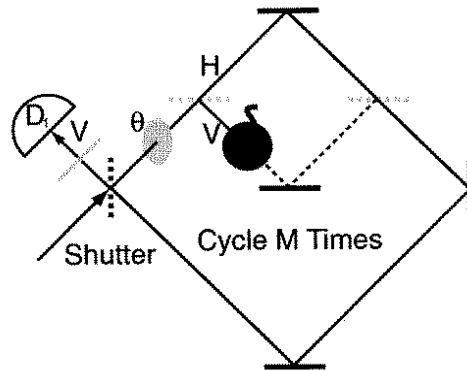
First we will consider a single pass through such that $M = 1$. The photon will be initially polarized in the horizontal direction, and so its initial state will be $|\Psi\rangle = |H\rangle$. Upon entering the apparatus, it passes through a θ wave plate, which rotates the polarization by an angle θ. (What is actually happening is that both horizontal and vertical polarizations are linear combinations of right and left circular polarization as discussed in class. The wave plates have different indices of refraction for right and left circular polarizations, and so induce a relative phase between the two. This the causes interference, and rotates a linear polarization.) The state of the system then becomes

$$|\Psi\rangle = \cos(\theta)\,|H\rangle + \sin(\theta)\,|V\rangle.$$

The two gray dashed lines represent polarizing beam splitters, which reflect one polarization, and transmit the other. The leftmost one reflects the vertically polarized light towards the bomb as indicated, while passing the horizontally polarized light. Now if there is no bomb both paths will induce the same phase shift, as the interferometer is perfectly balanced. Then at the second polarizing beam splitter the two beams will recombine so theta the state will be the same as before the first PBS, up to a global phase.

$$|\Psi\rangle_{\text{no bomb}} = \cos(\theta)\,|H\rangle + \sin(\theta)\,|V\rangle.$$

This state will then reflect off of the remaining mirrors, and exit towards the detector.

If, on the other hand there is a bomb, then all of the vertically polarized light shall be absorbed, ie. the photon will be detected if it is on this path, setting off the bomb with probability

$$P(\text{explode}) = |\langle V||\Psi\rangle|^2 = \sin^2(\theta)$$

If no explosion occurs (probability $\cos^2\theta$) then after the second PBS the photon must be horizontally polarized

$$|\Psi\rangle_{\text{bomb}} = |H\rangle$$

This state will then also propagate until it reaches the shutter and heads for the detector.

Now if we insert a polarizer in front of the detector, as indicated by the solid gray bar, and only let it pass light polarized at an angle $\pi/2 + \theta$, then the probability of

detecting a photon will be

$$P(\text{detection}) = P(\text{no explosion})|(\cos\theta \langle V| - \sin\theta \langle H|) |\Psi\rangle|^2.$$

Then we have that

$$(\cos\theta \langle V| - \sin\theta \langle H|) |\Psi\rangle_{\text{no bomb}} = 0$$

which can be easily shown by inserting the above result for this state. Then $P(\text{detection}|\text{no bomb}) = 0$, so we have the same result as in the original problem, that a detection indicates the presence of a bomb. The probability of detection with a bomb can then be calculated as

$$P(\text{detection}|\text{bomb}) = \cos^2(\theta)\sin^2(\theta).$$

This hasn't really gotten us anywhere, as this probability is still less than the probability of exploding the bomb $\sin^2\theta$. We still explode more often than we detect, so that if we send through photons until we get a definite result, the probability of exploding will be greater that 50%.

However, all is not lost. Up till now we have considered only classical repetition of the experiment, which we had hoped would boost our chances of detection. This method was obviously doomed to failure, as it increased our chances of exploding the bomb at the same rate as the detection probability. One possible way around this problem is to consider letting a photon run through the apparatus multiple times. The idea is that the bomb acts as a classical detector each time the photon passes through, and so the probability of explosion should increase only linearly (roughly) in the number of passes M, whether or not the same photon is used. On the other hand, the photon will not reach detector D_1 until completing all of its M passes, and so the probability amplitude for bomb detection should be linear in the number of passes, not the probability.

We can see this more rigorously by considering the state of the system after N passes. When there is no bomb present, the upper part of the apparatus (polarizing interferometer) has no effect, and so the final state will just be due to the N passes through the θ wave plate, an so will be

$$|\Psi(N)\rangle_{\text{no bomb}} = \cos(N\theta) |H\rangle + \sin(N\theta) |V\rangle.$$

In the presence of the bomb the vertically polarized component will always be absorbed on passing through the interferometer. The state of the system given that the bomb has not exploded will then be

$$|\Psi\rangle_{\text{bomb}} = |H\rangle,$$

independent of N. Using this the probability of the bomb exploding is

$$P(\text{explode}) = \sin^2\theta \sum_{i=0}^{N-1} \cos^2\theta.$$

Now if we insert a polarizer that passes angle $M\theta + \pi/2$, then the detection probability becomes

$$P(\text{detect}|\text{bomb}) = \sin^2(M\theta)\cos^{2M}(\theta)$$

if we take the limit $M \to \infty$, while holding $M\theta = \pi/2$, then this probability limits to 1, implying 100% detection. As a check we may take the same limit of the explosion probability to find that

$$P(\text{explode}) \to 1 - \cos^{2(M-1)}(\theta) \approx 0$$

Such that we can achieve near unit detection efficiency with vanishingly small probability of explosion.

Just as we initially hypothesized, the coherence of the system allowed us to increase the detection probability more quickly than the probability of explosion.

Problem 2 : The trace operator

Given a linear (bounded) operator $\hat{A}$

$$Tr(\hat{A}) = \sum_a \langle a | \hat{A} | a \rangle \quad \text{for an orthonormal basis } \{|a\rangle\}$$

(a) Given another orthonormal basis $\{|b\rangle\}$, $\sum_b |b\rangle\langle b| = 1$

$$Tr(\hat{A}) = \sum_a \sum_b \sum_{b'} \langle a|b\rangle\langle b|\hat{A}|b'\rangle\langle b'|a\rangle$$

$$= \sum_b \sum_{b'} \sum_a \langle b'|a\rangle\langle a|b\rangle\langle b|\hat{A}|b'\rangle$$

(Here I switched the order of summation & reordered the variable inner products. Remember they are <u>numbers</u>)

Now use $\sum_a |a\rangle\langle a| = 1$

$$Tr(\hat{A}) = \sum_b \sum_{b'} \langle b'| \left(\sum_a |a\rangle\langle a| \right) |b\rangle\langle b|\hat{A}|b'\rangle \quad \text{(linearity)}$$

$$= \sum_b \sum_{b'} \langle b'|b\rangle\langle b|\hat{A}|b'\rangle = \sum_b \sum_{b'} \delta_{b,b'} \langle b|\hat{A}|b'\rangle$$

$$\Rightarrow \boxed{Tr(\hat{A}) = \sum_b \langle b|\hat{A}|b\rangle} \Rightarrow \text{the trace of an operator is the sum of its diagonal matrix elements in any ~~orthogonal basis~~ orthonormal}$$

(25)

(i) $\text{Tr}(\hat{A}^\dagger) = \sum_i \langle i | \hat{A}^\dagger | i \rangle = \sum_i \langle i | \hat{A} | i \rangle^* = \left[\text{Tr}(\hat{A}) \right]^*$

(ii) $\text{Tr}(\hat{A} + \hat{B}) = \sum_i \langle i | (\hat{A} + \hat{B}) i \rangle = \sum_i \langle i | \hat{A} | i \rangle + \langle i | \hat{B} | i \rangle$

(iii) $\text{Tr}(\hat{A}\hat{B}) = \sum_i \langle i | \hat{A}\hat{B} | i \rangle = \sum_{ij} \langle i | \hat{A} | j \rangle \underbrace{\langle j |}_{\text{Complete set}} \hat{B} | i \rangle$

$\qquad = \sum_{ij} \langle j | \hat{B} | i \rangle \underbrace{\langle i |}_{\text{complete set}} \hat{A} | j \rangle = \sum_j \langle j | \hat{B}\hat{A} | j \rangle$

$\qquad = \text{Tr}(\hat{B}\hat{A})$

(iv) $\text{Tr}(|\phi\rangle\langle\psi|) = \sum_i \langle i | \phi \rangle \langle \psi | i \rangle = \sum_i \langle \psi | \underbrace{i \rangle \langle i |}_{\text{complete set}} \phi \rangle$

$\qquad = \langle \psi | \phi \rangle$

(v) $\text{Tr}(|\phi\rangle\langle\psi| \hat{A}) = \sum_i \langle i | \phi \rangle \langle \psi | \hat{A} | i \rangle$

$\qquad = \sum_i \langle \psi | \hat{A} | \underbrace{i \rangle \langle i |}_{\text{complete set}} \phi \rangle = \langle \psi | \hat{A} | \phi \rangle$

Note: Parts (iv) + (v) show that the trace turns outer products to inner products

(vi) ~~Let~~ Let $\{|a\rangle\}$ be the eigenvectors of $\hat{A}$
This is a basis for the space.

$\Rightarrow \text{Tr}(\hat{A}) = \sum_a \langle a | \hat{A} | a \rangle = \sum_a a \langle a | a \rangle = \sum_a a \checkmark$

This quantity is clearly basis independent

(2c) Restrictions?

We have the canonical commutator

$$[\hat{x}, \hat{p}] = i\hbar = \hat{x}\hat{p} - \hat{p}\hat{x}$$

What happens when we take the trace of both sides?

$$\text{Tr}(\hat{x}\hat{p}) - \text{Tr}(\hat{p}\hat{x}) = i\hbar \, \text{Tr}(\hat{1})$$

But $\text{Tr}(\hat{p}\hat{x}) = \text{Tr}(\hat{x}\hat{p}) \Rightarrow \text{Tr}(\hat{1}) = 0$!?

The resolution of this paradox stems from the fact that neither the trace of $\hat{x}$ nor $\hat{p}$ exist! This is because they act on an infinite dimensional Hilbert space and has infinite eigenvalues. Such operator are said to be __unbounded__

Note: Generally the commutator of two Hermitian operators will another operator, different from the identity

E.g. $[\hat{S}_x, \hat{S}_y] = i\hat{S}_z \Rightarrow \text{Tr}([\hat{S}_x, \hat{S}_y]) = 0 = \text{Tr}(\hat{S}_z)$
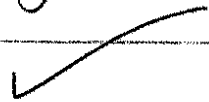
Problem 3: Spin ½ operators and eigenstates

$$\hat{S}_x = \frac{1}{2}\{|+_z\rangle\langle-|_z + |-\rangle_z\langle+|_z\}$$

$$\hat{S}_y = \frac{-i}{2}\{|+\rangle_z\langle-|_z - |-\rangle_z\langle+|_z\}$$

$$\hat{S}_z = \frac{1}{2}\{|+\rangle_z\langle+|_z - |-\rangle_z\langle-|_z\}$$

Three Cartesian components of spin angular momentum (in units of $\hbar$)

In matrix for in the basis $\{|+\rangle_z, |-\rangle_z\}$

$$\hat{S}_x \doteq \begin{bmatrix} \langle+_z|\hat{S}_x|+_z\rangle & \langle+_z|\hat{S}_x|-_z\rangle \\ \langle-_z|\hat{S}_x|+_z\rangle & \langle-_z|\hat{S}_x|-_z\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_y \doteq \begin{bmatrix} \langle+_z|\hat{S}_y|+_z\rangle & \langle+_z|\hat{S}_y|-_z\rangle \\ \langle-_z|\hat{S}_y|-_z\rangle & \langle-_z|\hat{S}_y|-_z\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\hat{S}_z \doteq \begin{bmatrix} \langle+_z|\hat{S}_z|+_z\rangle & \langle+_z|\hat{S}_z|-_z\rangle \\ \langle-_z|\hat{S}_z|+_z\rangle & \langle-_z|\hat{S}_z|-_z\rangle \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Recall the matricies $\hat{\sigma}_i = 2\hat{S}_i$ are

the Pauli matricies:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$$

(a) Eigenvalues and eigenvectors

We will work here with the spin matrices $\hat{S}_x, \hat{S}_y, \hat{S}_z$, though generally in the future it will be easier to work with the Pauli matrices

$$\hat{S}_x: \quad \det(\hat{S}_x - \lambda_x \hat{\mathbb{1}}) = \det \begin{bmatrix} -\lambda_x & \frac{1}{2} \\ \frac{1}{2} & -\lambda_x \end{bmatrix} = \lambda_x^2 - \frac{1}{4} = 0$$

$$\Rightarrow \boxed{\lambda_x = \pm \frac{1}{2}}$$

• $\underline{\lambda_x = +\frac{1}{2}} \Rightarrow \hat{S}_x |+_x\rangle = \frac{1}{2}|+_x\rangle = \frac{1}{2}\left(a_x^{(+)} |+_z\rangle + b_x^{(+)} |-_z\rangle\right)$

$$\Rightarrow \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_x^{(+)} \\ b_x^{(+)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a_x^{(+)} \\ a_x^{(+)} \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} a_x^{(+)} = \frac{1}{2} b_x^{(+)} \Rightarrow a_x^{(+)} = b_x^{(+)} \Rightarrow |+_x\rangle = a_x^{(+)}\left(|+_z\rangle + |-_z\rangle\right)$$
$$\text{(unnormalized)}$$

$$\langle +_x|+_x\rangle = |a_x^{(+)}|^2 \left(\langle +_z|+_z\rangle + \langle -_z|-_z\rangle\right) = 2|a_x^{(+)}|^2 = 1$$

$$\Rightarrow |a_x^{(+)}| = \frac{1}{\sqrt{2}}$$

Normalized eigenvector: $\boxed{|+_x\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle + |-_z\rangle\right)}$

• $\underline{\lambda_x = -\frac{1}{2}} \Rightarrow \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_x^{(-)} \\ b_x^{(-)} \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} a_x^{(-)} \\ b_x^{(-)} \end{bmatrix}$

$$\Rightarrow a_x^{(-)} = -b_x^{(-)} \Rightarrow |-_x\rangle = a_x^{(-)}\left(|+_z\rangle - |-_z\rangle\right)$$

Normalized eigenvector $\boxed{|-_x\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle - |-_z\rangle\right)}$

(a) continued

$\boxed{\hat{S}_y}$ $\det(\hat{S}_y - \lambda_y \hat{\mathbb{1}}) = \det\begin{bmatrix} -\lambda_y & -\frac{i}{2} \\ \frac{i}{2} & -\lambda_y \end{bmatrix} = \lambda_y^2 - \frac{1}{4} = 0$

$$\Rightarrow \boxed{\lambda_y = \cancel{} \pm \frac{1}{2}}$$

• $\underline{\lambda_y = +\frac{1}{2}} \Rightarrow \begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}\begin{bmatrix} a_y^{(+)} \\ b_y^{(+)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a_y^{(+)} \\ b_y^{(+)} \end{bmatrix} \Rightarrow a_y^{(+)} = -i b_y^{(+)}$

$$\Rightarrow \boxed{|+_y\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle - i|-_z\rangle\right)}$$

• $\underline{\lambda_y = -\frac{1}{2}} \Rightarrow \begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}\begin{bmatrix} a_y^{(-)} \\ b_y^{(-)} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} a_y^{(-)} \\ b_y^{(-)} \end{bmatrix} \Rightarrow a_y^{(+)} = +i b_y^{(+)}$

$$\Rightarrow \boxed{|-_y\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle + i|-_z\rangle\right)}$$

$\boxed{\hat{S}_z}$ Manefestly written in terms of eigenvectors

$|\pm_z\rangle$ with eigenvalues $\pm\frac{1}{2}$

These operators $\underline{are}$ Hermitian since $\hat{S}_i^\dagger = \hat{S}_i$

Eg: $\hat{S}_y^\dagger = -\frac{i^*}{2}\left(\left(|+_z\rangle\langle-_z|\right)^\dagger - \left(|-_z\rangle\langle+_z|\right)^\dagger\right) = \frac{i}{2}\left[|-_z\rangle\langle+_z| - |+_z\rangle\langle+_z|\right]$

$\quad\quad = \hat{S}_y$

The fact that the operators are Hermitian is reflected in the fact that the eigenvectors are $\underline{orthogonal}$ and the eigenvalues are $\underline{real}$

(b)

By completeness: $|+_x\rangle\langle+_x| + |-_x\rangle\langle-_x| = \hat{\mathbb{1}}$

$|+_z\rangle = |+_x\rangle\langle+_x|+_z\rangle + |-_x\rangle\langle-_x|+_z\rangle$

$= \langle+_z|+_x\rangle^* = \frac{1}{\sqrt{2}}$ $\langle+_z|-_x\rangle^* = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{|+_z\rangle = \frac{1}{\sqrt{2}}\left(|+_x\rangle + |-_x\rangle\right)}$

$|-_z\rangle = |+_x\rangle\langle+_x|-_z\rangle + |-_x\rangle\langle-_x|-_z\rangle$

$= \frac{1}{\sqrt{2}}$ $= -\frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{|-_z\rangle = \frac{1}{\sqrt{2}}\left(|+_x\rangle - |-_x\rangle\right)}$

⊗ Transformation matrix $|m_z\rangle = U|m_x\rangle$ $m = \pm$

$U \doteq \begin{bmatrix} \langle+_x|+_z\rangle & \langle+_x|-_z\rangle \\ \langle-_x|+_z\rangle & \langle+_x|-_z\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$U^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}$

$UU^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

✓

U is unitary

(c) Express $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in basis $\{|+_x\rangle, |-_x\rangle\}$

We are given

$$\hat{S}_j = \sum_{m_z, m_z'} |m_z\rangle\langle m_z| \hat{S}_j |m_z'\rangle\langle m_z'| \qquad m_z, m_z' = + \text{ or } -$$

We seek $$\langle m_x|\hat{S}_j|m_x'\rangle = \sum_{m_z, m_z'} U^\dagger_{m_x, m_z} \langle m_z|\hat{S}_j|m_z'\rangle U_{m_z', m_x'}$$

where $U_{m_z, m_x} = \langle m_z|m_x\rangle$

Here I will write out the matricies

$$S \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$\hat{S}_x$ is basis $\{|\pm_x\rangle\} = U^\dagger \quad \hat{S}_x$ in basis $\{|\pm_z\rangle\} \quad U$

$$\Rightarrow S_x \doteq \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \Rightarrow \boxed{\hat{S}_x = \frac{1}{2}|+_x\rangle\langle +_x| - \frac{1}{2}|-_x\rangle\langle -_x|}$$

Diagonal in basis of eigenstates

$$\hat{S}_y \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{S}_y = \frac{i}{2}\{|+_x\rangle\langle -_x| - |\Phi_x\rangle\langle +_x|\}}$$

(Next Page)

$$S_z \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \quad \boxed{\hat{S}_z = \frac{1}{2}\left\{ |+_x\rangle\langle -_x| + |-_x\rangle\langle +_x| \right\}}$$
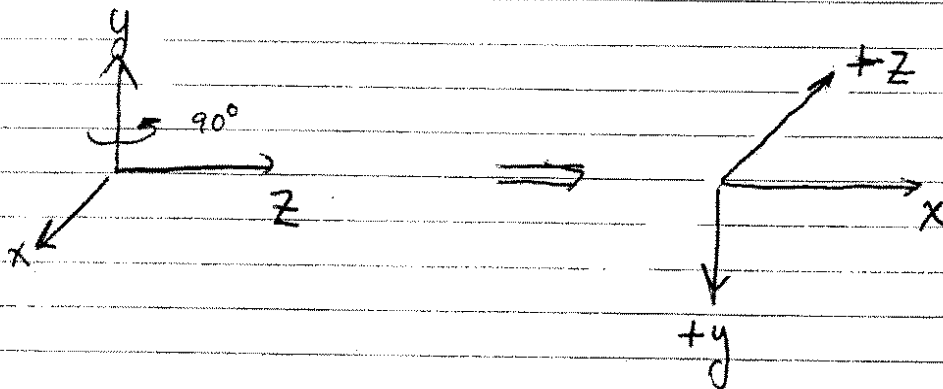
Note: The transformation we are making is like a rotation about the $\hat{y}$-axis by $90°$

Thus we expect

$$\hat{S}_z \rightarrow \hat{S}_x \qquad \text{In the new representation}$$
$$\hat{S}_x \rightarrow \hat{S}_z$$

The surprise is that $\hat{S}_y \rightarrow -\hat{S}_y$

this is because the transformation is really a rotation together with inversion about y-axis to preserve the right hand rule

Rotation about y-axis by $90°$

and $y \rightarrow -y$