Physics 521

Problem Set # 2 Solutions

Problem 1: Functions of an operator

Given an analytic function $f(x) = \sum_{n\geq 0}^{\infty} f_n x^n$

we define the operator $f(\hat{A}) = \sum_{n=0}^{\infty} f_n \hat{A}^n$

(a) In its eigen-decomposition $\hat{A} = \sum_a a \hat{\mathbb{P}}_a$

where implicitly, I assume $\mathbb{P}_a = |a\rangle\langle a|$ (one D)

Aside $\left\{ \begin{array}{l} : \ \mathbb{P}_a^m \mathbb{P}_{a'}^n \ \hat{\mathbb{P}}_a \delta_{aa'} \text{, Since this is Projection.} \\ \Rightarrow \ \hat{A}^n = \sum_a a^n \hat{\mathbb{P}}_a \end{array} \right\}$

$$\therefore \ f(\hat{A}) = \sum_a \left(\sum_{n=0}^{\infty} f_n a^n \right) \hat{\mathbb{P}}_a = \sum_a f(a) \hat{\mathbb{P}}_a$$

Thus we can think of a function of a diagonalizable operator as taking function of its eigenvalues.

(b) Let $\hat{A}$ be anti-Hermitian, and define $f(\hat{A}) = e^{\hat{A}}$

Note: $\hat{A}$ must be normal $\Rightarrow [\hat{A}, \hat{A}^{+}] = 0$

Now $f(\hat{A})^{+} = \left(\sum_n \frac{1}{n!} \hat{A}^n \right)^{+} = \sum_n \frac{1}{n!} (A^{+})^n = e^{\hat{A}^{+}}$

$= e^{-\hat{A}}$

$$\therefore \ f(\hat{A}) \cdot f(\hat{A})^{+} = e^{\hat{A}} \cdot e^{-\hat{A}} = e^{\hat{A} + (-\hat{A})} = e^{0} = \mathbb{1}$$

$\because [\hat{A}, -\hat{A}] = 0$

~~The goal~~

S~~how that~~

(c) let $\hat{H}$ has the following eigen decomposition

$$\hat{H} = \sum_n E_n |E_n\rangle\langle E_n|$$

$$\hat{A} = i\hat{H} = \sum_n iE_n |E_n\rangle\langle E_n|$$

$$f(\hat{A}) = e^{\hat{A}} = e^{i\hat{H}} = \sum_n e^{iE_n} |E_n\rangle\langle E_n|$$

(d) The goal

Show that $U B U^{\dagger} = e^{\hat{A}} B e^{-\hat{A}}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{A}, \hat{B}]^{(n)}$$

where $[\hat{A}, \hat{B}]^{(n)} \equiv [\hat{A}, [A, B]^{(n-1)}]$

$$= \underbrace{[\hat{A}, [\hat{A}, [\hat{A} \cdots}_{n \text{ times}} [\hat{A}, \hat{B}]]$$

$$\Rightarrow e^{\hat{A}} B e^{-\hat{A}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{n!\, m!} \hat{A}^n \hat{B} \hat{A}^m$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!\, m!} \hat{A}^{n-m} \hat{B} \hat{A}^m$$ (Grouping terms of given total power of $\hat{A}$)

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{(n-m)!\, m!} (-1)^m \hat{A}^{n-m} \hat{B} \hat{A}^m$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \hat{A}^{n-m} \hat{B} \hat{A}^m$$

Binomial coefficiant

n-chouse m

Aside: Note if $[\hat{A},\hat{B}]=0$ then

$$[\hat{A},\hat{B}]^{(n)} = (\hat{A}-\hat{A})^n \hat{B}$$

$$= \delta_{n,0} \text{ as expected}$$

We will just show first few terms

$n=0$ $\quad [\hat{A},\hat{B}]^{(0)} = \hat{B} \overset{?}{=} \sum_{m=0}^{0} \binom{0}{m}(-1)^m \hat{A}^{-m}\hat{B}\hat{A}^m$

$$= \binom{0}{0}\hat{B} = \hat{B} \checkmark$$

$n=1$ $\quad [\hat{A},\hat{B}]^{(1)} = \hat{A}\hat{B}-\hat{B}\hat{A} \overset{?}{=} \sum_{m=0}^{1} \binom{1}{m}(-1)^m \hat{A}^{1-m}\hat{B}\hat{A}^m$

$$= \binom{1}{0}\hat{A}\hat{B} - \binom{1}{1}\hat{B}\hat{A}$$

$$= \hat{A}\hat{B}-\hat{B}\hat{A} \checkmark$$

$n=2$ $\quad [\hat{A},\hat{B}]^{(2)} = [\hat{A},[\hat{A},\hat{B}]]$

$$= \hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} - (\hat{A}\hat{B}\hat{A} - \hat{B}\hat{A}^2)$$

$$= \hat{A}^2\hat{B} - 2\hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2$$

$$\overset{?}{=} \sum_{m=0}^{2} \binom{2}{m}(-1)^m \hat{A}^{2-m}\hat{B}\hat{A}^m$$

$$= \binom{2}{0}\hat{A}^2\hat{B} - \binom{2}{1}\hat{A}\hat{B}\hat{A} + \binom{2}{2}\hat{B}\hat{A}^2$$

$$= \hat{A}^2\hat{B} - 2\hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2 \checkmark$$

$n=3$, $\quad [\hat{A},\hat{B}]^{(3)} = [\hat{A},[\hat{A},\hat{B}]^{(2)}] = \hat{A}^3\hat{B} - 2\hat{A}^2\hat{B}\hat{A} + \hat{A}\hat{B}\hat{A}^2$

$$- (\hat{A}^2\hat{B}\hat{A} - 2\hat{A}\hat{B}\hat{A}^2 + \hat{B}\hat{A}^3)$$

$$= \hat{A}^3\hat{B} - 3\hat{A}^2\hat{B}\hat{A} + 3\hat{A}\hat{B}\hat{A}^2 - \hat{B}\hat{A}^3$$

$$\stackrel{?}{=} \sum_{m=0}^{3} \binom{3}{m} (-1)^m \hat{A}^{3-m} \hat{B} \hat{A}^m$$

$$= \binom{3}{0} \hat{A}^3 \hat{B} - \binom{3}{1} \hat{A}^2 BA + \binom{3}{2} \hat{A} B\hat{A}^2 - \binom{3}{3} \hat{B}\hat{A}^3$$

$$= \hat{A}^3 B - 3\hat{A}^2 B\hat{A} + 3\hat{A}B\hat{A}^2 - \hat{B}\hat{A}^3$$

This holds for all(n)

Problem 2 : Polar Decomposition

Consider an operator $\hat{M}$ and define $\hat{R}_L = \sqrt{MM^+}$

and $\hat{R}_R = \sqrt{\hat{M}^+\hat{M}}$. Note $\hat{M}^+M$ and $\hat{M}\hat{M}^+$ are

Positive operators, Since $\forall |\psi\rangle \Rightarrow \begin{cases} \langle\psi|\hat{M}^+M|\psi\rangle = \|\hat{M}|\psi\rangle\|^2 \geqslant 0 \\ \langle\psi|\hat{M}\hat{M}^+|\psi\rangle = \|\hat{M}^+|\psi\rangle\|^2 \geqslant 0 \end{cases}$

$\Rightarrow$ $\hat{R}_R$ and $\hat{R}_L$ are Positive Hermitian operator

To see this express $\hat{M}^+M = \sum_{\lambda R} \mu_R |\mu_R\rangle\langle\mu_R|$ has eigen-decomp)

(ie $\hat{M}^+\hat{M}$ is Hermitian and thus

Since $\mu_R \geqslant 0$ $\quad \sqrt{\mu_R} \geqslant 0$

$\Rightarrow$ $\hat{R}_R = \sum_R \sqrt{\mu_R} |\mu_R\rangle\langle\mu_R|$ is Positive Hermitian

The same holds for $\hat{R}_L$

We further assume $\hat{M}$ is invertible $\Rightarrow$ no zero eigenvalues

$\Rightarrow$ $\hat{R}_L$ and $\hat{R}_R$ are invertible

Now define $\hat{U}_L \hat{R}_L^{-1} \hat{M}$ and $\hat{U}_R = \hat{M}\hat{R}_R^{-1}$

we seek to show that $\hat{U}_L = \hat{U}_R$ and it is a unitary operator.

Let us first show that $\hat{U}_L$ is Unitary

$$\hat{U}_L^\dagger = \hat{M}^\dagger (R_L^{-1})^\dagger = M^\dagger R_L^{-1} \longrightarrow \text{Hermitian}$$

$$\Rightarrow U_L U_L^\dagger = \frac{1}{\sqrt{\hat{M}\hat{M}^\dagger}} \hat{M}\hat{M}^\dagger \frac{1}{\sqrt{\hat{M}\hat{M}^\dagger}} = \frac{1}{\hat{M}\hat{M}^\dagger} \hat{M}\hat{M}^\dagger$$

$$= \mathbb{1}$$

In finite dim $\Rightarrow$ $U_L^\dagger U_L = \mathbb{1}$ also

The same follows for $\hat{U}_R$

Now show $\hat{U}_R = \hat{U}_L = \hat{U}$

If ~~done~~ true $\hat{R}_L \hat{U} \overset{?}{=} \hat{U}\hat{R}_R$

$$\Rightarrow U^\dagger R_L \hat{U} \overset{?}{=} \hat{R}_R$$

To show last line, multiply both sides by their adjoints:

$$R_R^\dagger \hat{R}_R = \hat{R}_R^2 \overset{?}{=} \hat{U}^\dagger R_L \hat{U}\, U^\dagger \hat{R}_L^\dagger \hat{U}$$

$$\Rightarrow \hat{R}_R^2 \overset{?}{=} \hat{U}^\dagger \hat{R}_L^2 \hat{U}$$

$$\Rightarrow \hat{M}^\dagger \hat{M} \overset{?}{=} \left(\hat{M}^\dagger \frac{1}{\hat{R}_L} R_L^2 \frac{1}{\hat{R}_L} \hat{M} \right)$$

$$\hat{M}^\dagger \hat{M} \overset{\checkmark}{=} \hat{M}^\dagger M$$

Thus given $\hat{M}$ with $\hat{R}_R = \sqrt{\hat{M}^\dagger M}$ invertibles
$\hat{R}_L = \sqrt{\hat{M}\hat{M}^\dagger}$

$\exists$ a unitary s.t

$$\hat{M} = \hat{R}_L U = \hat{U}\hat{R}_R$$

This is clearly an operator generalization of the polar decomposition for complex numbers. $z = re^{i\theta}$ $\quad r = \sqrt{zz^*} \geq 0$

$$e^{i\theta} = z/r$$

(b) Given a normal operator $\hat{M} = \sum_{\lambda} \lambda |\lambda\rangle\langle\lambda|$

$\lambda = |\lambda| e^{i\theta_\lambda}$

$$\hat{R}_L = \hat{R}_R = \sum_{\lambda} |\lambda| \, |\lambda\rangle\langle\lambda|$$

$$\hat{U} = \sum_{\lambda} e^{i\theta_\lambda} |\lambda\rangle\langle\lambda|$$

$$\Rightarrow \hat{M} = \sum_{\lambda} |\lambda| e^{i\theta_\lambda} |\lambda\rangle\langle\lambda|$$

$[\hat{R}, \hat{U}] = 0$ for nomal operator.

(c) we need to find the Polar decomposition of

$$\hat{M} = \mathbb{1} + \frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}$$

This operator may be represented as

$$\hat{M} \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Then we can find

$$R_L^2 = \hat{M}\hat{M}^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

We then need to find the square root of this matrix to get $\hat{R}_L$. Just guess'ing a matrix that squares to this one is insufficient, as there are many such matrices. We must use what we know from Problem one to take square root.

Since $\hat{R}_L^2 = \hat{M}\hat{M}^\dagger$ is hermitian, we use following result

$$f(\hat{H}) = \sum_{h} f(h) |h\rangle\langle h|$$

first we find eigenvalues of $\hat{R}_L^2$, which are

$$\frac{3 \pm \sqrt{5}}{2}$$

We also need the eigenvectors, which we can find by solving the equation

$$\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Solving gives $\quad b = \frac{1 \pm \sqrt{5}}{2} a$

we also require normalization so that $a^2 + b^2 = 1$

$$a^2 = \frac{2}{5 \pm \sqrt{5}} = \frac{1}{\sqrt{5}} \frac{2}{\sqrt{5} \pm 1}$$

Now we just take positive square roots of eigenvalue.

$$\frac{3 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2} \right)^2$$

$$\lambda_{\pm} = \frac{\sqrt{5} \pm 1}{2} \quad \text{(Positive roots)}$$

eigen vectors are $\quad |\lambda_\pm\rangle \Rightarrow \frac{1}{\sqrt[4]{5}} \frac{1}{\sqrt{\lambda_\pm}} \begin{pmatrix} 1 \\ \pm \lambda_\pm \end{pmatrix}$

We have $\quad \hat{R}_L = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-|$

This can be worked out in matrix form

$$\hat{R}_L \rightarrow \frac{1}{\sqrt{5}} \left[\begin{pmatrix} 1 & \lambda_+ \\ \lambda_+ & \lambda_+^2 \end{pmatrix} + \begin{pmatrix} 1 & -\lambda_+ \\ -\lambda_- & \lambda_-^2 \end{pmatrix} \right]$$

$$\rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Matrix can now be inverted

$$R_L^{-1} \rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

then we can find

$$\hat{U} = R_L^{-1}\hat{M} \rightarrow \frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$$

$$\therefore\ \hat{M} = \hat{R}_L\hat{U} = \left[\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\right]\left[\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\right]$$

For right polarization $\hat{R}_R = \hat{U}^{+}\hat{R}_L\hat{U}^{*} \rightarrow \frac{1}{\sqrt{5}}\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}$

$$\hat{M} = \hat{U}\hat{R}_R \rightarrow \left[\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\right]\left[\frac{1}{\sqrt{5}}\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}\right]$$

Problem 3: The angular momentum operator

(a) $\hat{L}_i = \sum_{j,k}\epsilon_{ijk}\hat{x}_j\hat{P}_k$

$$[\hat{L}_i, \hat{L}_j] = \left[\sum_{a,b}\epsilon_{iab}\hat{x}_a\hat{P}_b, \sum_{c,d}\epsilon_{jcd}\hat{x}_c\hat{P}_d\right]$$

$$= \sum_{a,b}\sum_{c,d}\epsilon_{i,a,b}\epsilon_{j,c,d}[\hat{x}_a\hat{P}_b, \hat{x}_c\hat{P}_d]$$

$$= \sum_{a,b,c,d}\epsilon_{iab}\epsilon_{jcd}\left\{\hat{x}_a[\hat{P}_b, \hat{x}_c\hat{P}_d] + [\hat{x}_a, \hat{x}_c\hat{P}_d]\hat{P}_b\right\}$$

$$= \sum_{a,b,c,d}\epsilon_{iab}\epsilon_{jcd}\left\{\hat{x}_a\hat{x}_c\overbrace{[\hat{P}_b, \hat{P}_d]}^{\to 0} + \hat{x}_a[\hat{P}_b, \hat{x}_c]\hat{P}_d + \hat{x}_c[\hat{x}_a, \hat{P}_d]\hat{P}_b + \overbrace{[\hat{x}_a, \hat{x}_c]}^{0}\hat{P}_d\hat{P}_b\right\}$$

$$= \sum_{a,b,c,d}\epsilon_{iab}\epsilon_{jcd}\left\{-i\delta_{bc}\hbar\,\hat{x}_a\hat{P}_d + i\delta_{ad}\hbar\,\hat{x}_c\hat{P}_b\right\}$$

$$= i\hbar\left[\sum_{a,b,c,d}\left(-\epsilon_{iab}\epsilon_{jcd}\delta_{bc}\hat{x}_a\hat{P}_d + \epsilon_{iab}\epsilon_{jcd}\delta_{ad}\hat{x}_c\hat{P}_b\right)\right]$$

$$= i\hbar\left[\sum_{a,b,d}\underbrace{\epsilon_{bia}\epsilon_{bjd}}_{\text{exchanged } c \,\&\, j \Rightarrow -\text{ve sign} \,\&\, b=c}\hat{x}_a\hat{P}_d - \sum_{abc}\epsilon_{aib}\epsilon_{ajc}\delta\,\hat{x}_c\hat{P}_b\right]$$

$$\boxed{\text{Now, } \sum_i \epsilon_{iab}\,\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \Big[\sum_{a,d} (\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj})\hat{x}_a\hat{P}_d - \sum_{b,c}(\delta_{ij}\delta_{bc} - \delta_{ic}\delta_{bj})\hat{x}_c\hat{P}_b\Big)$$

$$= i\hbar\Big[\sum_a(\delta_{ij}\hat{x}_a\hat{P}_a) - \hat{x}_j\hat{P}_i - \sum_b \delta_{ij}\hat{x}_b\hat{P}_b + \hat{x}_i\hat{P}_j\Big)$$

$$= i\hbar\,[\hat{x}_i\hat{P}_j - \hat{x}_j\hat{P}_i]$$

$$= i\hbar\sum_k \epsilon_{ijk}\hat{L}_k$$

(b) $$[\hat{L}_i, \hat{x}_j] = \cancel{\Big[\sum_{j,k,l}\epsilon_{ikl}}$$

$$= \Big[\sum_{k,l}\epsilon_{ikl}\hat{x}_k\hat{P}_l, \hat{x}_j\Big]$$

$$= \sum_{k,l}\epsilon_{ikl}\hat{x}_k[\hat{P}_l, \hat{x}_j]$$

$$= \sum_{k,l} -i\hbar\,\delta_{lj}\,\epsilon_{ikl}\hat{x}_k$$

$$= \sum_k \epsilon_{ijk}\, i\hbar\, \hat{x}_k$$

$$[\hat{L}_i\ \hat{P}_j] = \Big[\sum_{kl}\epsilon_{ikl}\hat{x}_k\hat{P}_l, \hat{P}_j\Big]$$

$$= \sum_{kl}\epsilon_{ikl}\, i\hbar\,\delta_{kj}\hat{P}_l$$

$$= i\hbar\sum_l \epsilon_{ijl}\hat{P}_l = i\hbar\sum_k \epsilon_{ijk}\hat{P}_k$$

$$[\hat{L}_i, \hat{r}^2] = \left[\sum_{j,k} \epsilon_{ijk} \hat{x}_j \hat{p}_k, \sum_{l} \hat{x}_l \hat{x}_l\right]$$

$$= \sum_{j,k,l} \epsilon_{ijk} \hat{x}_j \left\{ x_l [\hat{p}_k, \hat{x}_l] + [\hat{p}_k, \hat{x}_l] \hat{x}_l \right\}$$

$$= \sum_{jkl} \epsilon_{ijk} \left(-i\hbar \delta_{kl} \hat{x}_j x_l + -i\hbar \delta_{kl} \hat{x}_l \right\}$$

$$= -i\hbar \sum_{jkl} (\epsilon_{ijl} + \epsilon_{ijl}) \hat{x}_j \hat{x}_l$$

$$= -2i\hbar \sum_{jl} \epsilon_{ijl} \hat{x}_j \hat{x}_l$$

also can be written as $-2i\hbar(\vec{\hat{x}} \times \vec{\hat{x}})_i = 0$

$$\cancel{= -i\hbar \sum_{jl} \epsilon_{ijl} (\hat{x}_j \hat{x}_l + \hat{x}_l \hat{x}_j)}$$

$$= -i\hbar \sum_{jl} (\epsilon_{ijl} \hat{x}_j \hat{x}_l + \epsilon_{ilj} \hat{x}_l \hat{x}_j)$$

$$= -i\hbar \sum_{jl} \underbrace{(\epsilon_{ijl} + \epsilon_{ilj})}_{0} \hat{x}_j \hat{x}_l$$

$$= 0$$

$$[\hat{L}_i, \hat{p}^2] = \left[\sum_{jk} \epsilon_{ijk} \hat{x}_j \hat{p}_k, \sum_{l} \hat{p}_l \hat{p}_l\right]$$

$$= \sum_{j,kl} \epsilon_{ijk} \left(2 [\hat{x}_j, \hat{p}_l] \hat{p}_l \hat{p}_k \right]$$

$$= 2i\hbar \sum_{jkl} \delta_{jl} \epsilon_{ijk} \hat{p}_l \hat{p}_k$$

$$= 2i\hbar \sum_{jk} \epsilon_{ijk} \hat{p}_j \hat{p}_k$$

$$= 2i\hbar (\vec{\hat{p}} \times \vec{\hat{p}})_i = 0$$

$$[\hat{L}_i, \hat{L}^2] = \cancel{\left[\hat{L}_i, \sum_{l} L_l L_l\right]} \; [\hat{L}_i, \sum_j L_j L_j]$$

$$[\hat{L}_i, \hat{L}_j] \hat{L}_j + \hat{L}_j [\hat{L}_i, \hat{L}_j]$$

$$= \sum_{j} [\hat{L}_i, \hat{L}_j] \hat{L}_j + \ldots$$

$$= \sum_{j} \left(\sum_{k} i\hbar \epsilon_{ijk} \hat{L}_k \hat{L}_j + \sum_{k} i\hbar \epsilon_{ijk} \hat{L}_j \hat{L}_k \right)$$

$$= i\hbar \sum_{j,k} \epsilon_{ijk} \left(\hat{L}_k \hat{L}_j + \hat{L}_j \hat{L}_k \right)$$

$$= i\hbar \left(\sum_{jk} \epsilon_{ijk} \hat{L}_k \hat{L}_j + \sum_{jk} \epsilon_{ijk} \hat{L}_j \hat{L}_k \right)$$

$$= i\hbar \left(\sum_{jk} \epsilon_{ikj} \hat{L}_j \hat{L}_k + \sum_{jk} \epsilon_{ijk} \hat{L}_j \hat{L}_k \right)$$

$$= i\hbar \sum_{jk} \hat{L}_j \hat{L}_k \underbrace{(\epsilon_{ikj} + \epsilon_{ijk})}_{0}$$

$$= 0$$

(c) We know that

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

So $\Delta J_x \Delta J_y \geq \frac{1}{2} |\langle [J_x, J_y] \rangle|$

$$\geq \frac{\hbar}{2} |\langle J_z \rangle|$$

$S_x|+z\rangle = \frac{\hbar}{2}|-z\rangle$

$S_x|-z\rangle = \frac{\hbar}{2}|+z\rangle$

$S_y|+z\rangle = \frac{i\hbar}{2}|+z\rangle$

$S_y|-z\rangle = \frac{-i\hbar}{2}|+z\rangle$

(d) $$\Delta S_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$= \langle +z| S_x^2 |+z\rangle - \langle +z|S_x|+z\rangle^2$$

$$= \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$$\Delta S_x = \frac{\hbar}{2}$$

$$\Delta S_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2$$

$$= \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$\Delta S_y = \frac{\hbar}{2}$, $\Delta S_x \Delta S_y = \frac{\hbar^2}{4}$

$$\langle S_z \rangle = \langle +z|S_z|+z\rangle = \frac{\hbar}{2}$$

$$\Delta S_x \Delta S_y = \frac{\hbar^2}{4} = \frac{\hbar}{2} |\langle S_z \rangle| = \frac{\hbar^2}{4}$$