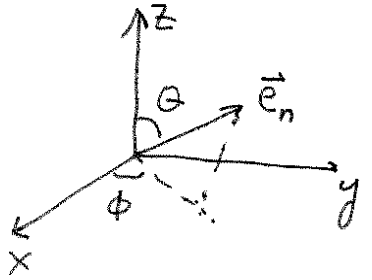
Physics 521: Quantum I: Problem Set #3 Solutions

Problem 1: Measurement on a two-state system

Consider direction in 3D space



$$\vec{e}_n = \cos\theta \vec{e}_z + \sin\theta(\cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y)$$

We define the spin operator along the direction $\vec{e}_n$

$$\hat{S}_n \equiv \hat{\vec{S}} \cdot \vec{e}_n = \sin\theta(\cos\phi\, \hat{S}_x + \sin\phi\, \hat{S}_y) + \cos\theta\, \hat{S}_z$$

Where
$$\hat{S}_x = \frac{\hbar}{2}\left(|+_z\rangle\langle -_z| + |-_z\rangle\langle +_z|\right) \equiv \frac{\hbar}{2}\hat{\sigma}_x$$
$$\hat{S}_y = -i\frac{\hbar}{2}\left(|+_z\rangle\langle -_z| - |-_z\rangle\langle +_z|\right) \equiv \frac{\hbar}{2}\hat{\sigma}_y$$
$$\hat{S}_z = \frac{\hbar}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right) \equiv \frac{\hbar}{2}\hat{\sigma}_z$$

Recall $\hat{\sigma}_x |\pm\rangle_z = |\mp_z\rangle$, $\hat{\sigma}_y |\pm\rangle_z = \pm i |\mp_z\rangle$, $\hat{\sigma}_z |\pm_z\rangle = \pm |+_z\rangle$

Now define $|+_n\rangle \equiv \cos\frac{\theta}{2}|+_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-_z\rangle$

The parameters θ and ϕ completely specify any pure state in for spin ½:
- θ determines relative probability of $|+_z\rangle$ vs. $|-_z\rangle$
- ϕ determines relative phase

We seek to show: $\hat{S}_n |+_n\rangle =$

Proof:

$$\hat{S}_n |+_n\rangle = \cos\frac{\theta}{2}\hat{S}_n |+_z\rangle + e^{i\phi}\sin\frac{\theta}{2}\hat{S}_n |-_z\rangle$$

$$= \cos\frac{\theta}{2}\left[\sin\theta\left(\cos\phi\ \hat{S}_x |+_z\rangle + \sin\phi\,\hat{S}_y |+_z\rangle\right) + \cos\theta\,\hat{S}_z |+_z\rangle\right]$$

$$+ e^{i\phi}\sin\frac{\theta}{2}\Big[\quad \Downarrow \quad |-_z\rangle \qquad \Downarrow \quad |-_z\rangle \qquad \Uparrow \quad |-_z\rangle\Big]$$

$$= \frac{\hbar}{2}\cos\frac{\theta}{2}\left[\sin\theta\,(\cos\phi + i\sin\phi)\,|-_z\rangle + \cos\theta\,|+_z\rangle\right]$$

$$+ \frac{\hbar}{2}e^{i\phi}\sin\frac{\theta}{2}\left[\sin\theta\,(\cos\phi - i\sin\phi)\,|+_z\rangle - \cos\theta\,|-_z\rangle\right]$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right)|+_z\rangle$$

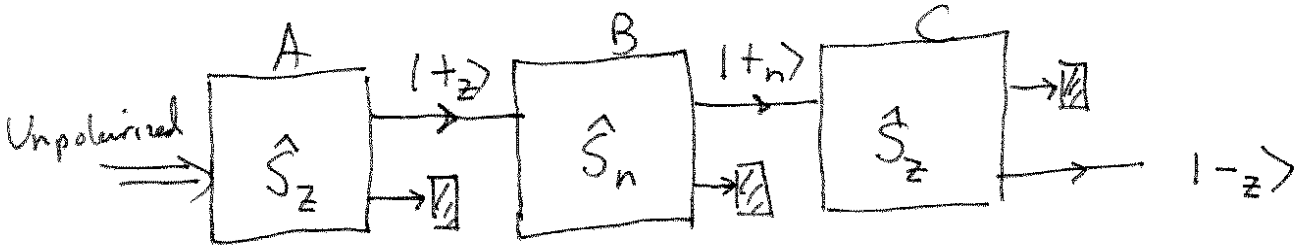
$$+ \frac{\hbar}{2}\left(\cos\frac{\theta}{2}\sin\theta - \sin\frac{\theta}{2}\cos\theta\right)e^{i\phi}|-_z\rangle$$

$$= \frac{\hbar}{2}\left[\cos\left(\theta - \frac{\theta}{2}\right)|+_z\rangle + e^{i\phi}\sin\left(\theta - \frac{\theta}{2}\right)|-_z\rangle\right]$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2}|+_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-_z\rangle\right)$$

$$\Rightarrow \boxed{\hat{S}_n |+_n\rangle = \frac{\hbar}{2}|+_n\rangle} \quad \text{Whew!}$$

(b) Now consider the following gedanken experiment



cascaded Stern-Gerlach devices

Probability of finding $|-_z\rangle$ after apparatus C

$$P^C(-_z) = P^C(-_z|+_n)\, P^B(+_n|+_z)$$
$$= \underbrace{|\langle -_z|+_n\rangle|^2}\ \underbrace{|\langle +_n|+_z\rangle|^2}$$

probability that C finds $|-_z\rangle$ given pure state $|+_n\rangle$

probability that B finds $|+_n\rangle$ given pure state $|+_z\rangle$

(We have renormalized the result of A)

Aside: From part (a) $|\langle -_z|+_n\rangle|^2 = \sin^2\frac{\theta}{2}$

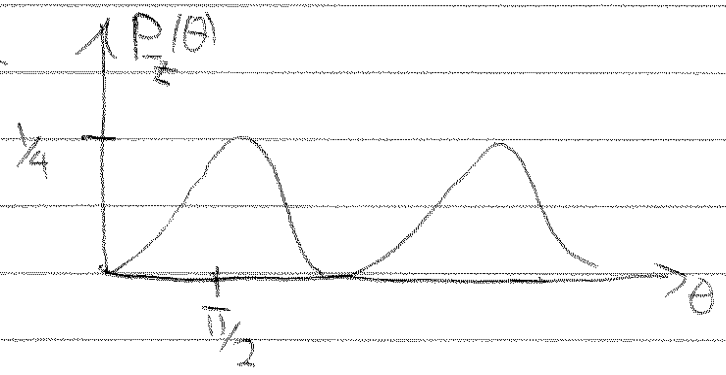
$|\langle +_n|+_z\rangle|^2 = \cos^2\frac{\theta}{2}$

$$\Rightarrow \quad = \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2$$

$$\Rightarrow \boxed{P^C(-_z) = \frac{1}{4}\sin^2\theta}$$

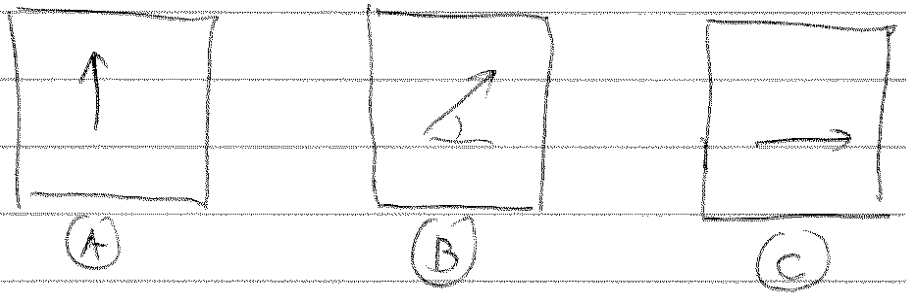
(c) How should we orient the middle apparatus to maximized output?

Sketch



⇒ [Orient $\vec{e}_n$ in the xy plane]

This problem is analogous to the optics problem whereby we have three polarizers:

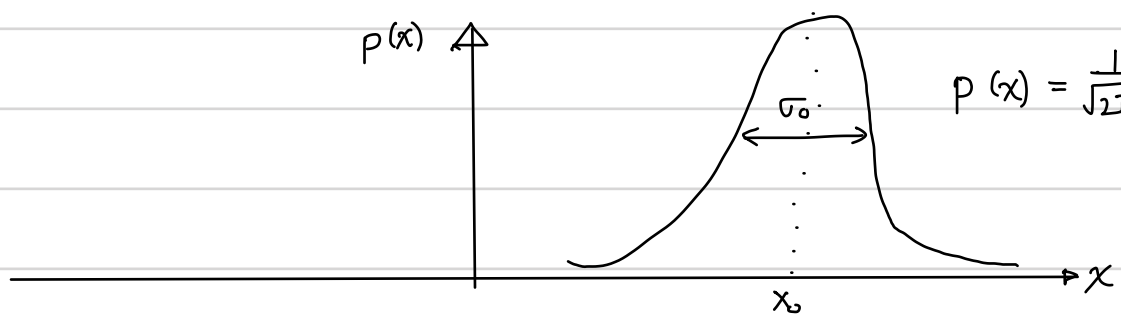


Without polarizer (B), (A) and (C) are orthogonal and nothing passes (C). However by inserting the middle polarizer and maximizing the passage of intensity we can transmit 1/4 of the intensity which passes (A)

Problem 2: Beyond Projective Measurements

(a) Bayes rule with Gaussians.

We are trying to determine the position of a particle along one dimension. Our "prior" probability distribution (best guess before the measurement is made) is a Gaussian



$$p(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-x_0)^2}{2\sigma_0^2}}$$

Note: This distribution has the following properties:

(1) It is normalized: $\int_{-\infty}^{\infty} dx\ p(x) = 1$

(2) $\bar{x} = \int_{-\infty}^{\infty} x\, p(x)\, dx = x_0$ mean

(3) $\Delta x^2 = \overline{x^2} - \bar{x}^2 = \int_{-\infty}^{\infty} x^2 p(x)\, dx - x_0^2 = \sigma_0^2$ (variance)

These are integrals you can prove, but once you do it, you need never do it again.

The measuring apparatus has finite resolution. The conditional probability density that my detector will register the position y, given that the true probability was x is

$$P(y|x) = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(y-x)^2}{2\Delta^2}}$$ (Centered on the true position, but "smeared" over Δ)

According to Bayes rule $p(x|y) = \frac{p(y|x)\, p(x)}{p(y)}$ (posterior; conditional; prior; renormalization)

Here $p(y)$ is the "marginal" distribution that ensures that the posterior is normalized

$$p(y) = \int_{-\infty}^{\infty} dx\ \underbrace{p(x,y)}_{\text{joint probability}} = \int_{-\infty}^{\infty} dx\ p(y|x)\, p(x)$$

Let us first calculate $p(x,y) = p(y|x)\, p(x) = \frac{1}{2\pi\sigma_0\Delta} \exp\left\{ -\frac{(x-x_0)^2}{2\sigma_0^2} - \frac{(y-x)^2}{2\Delta^2} \right\}$

Aside: $\frac{(x-x_0)^2}{\sigma_0^2} + \frac{(y-x)^2}{\Delta^2} = \frac{x^2 - 2xx_0 + x_0^2}{\sigma_0^2} + \frac{y^2 - 2yx + x^2}{\Delta^2} = \frac{x^2}{\sigma'^2} - 2x\,A(y) + B(y)$

where $\frac{1}{\sigma'^2} = \frac{1}{\sigma_0^2} + \frac{1}{\Delta^2}$, $A(y) = \frac{x_0}{\sigma_0^2} + \frac{y}{\Delta^2}$, $B(y) = \frac{x_0^2}{\sigma_0^2} + \frac{y^2}{\Delta^2}$

"Complete the square" $\frac{x^2}{\sigma'^2} - 2x\,A(y) = \frac{(x-x')^2}{\sigma'^2} - \frac{x'^2}{\sigma'^2}$, where $x' = \sigma'^2 A(y)$

$\Rightarrow x' = \left(\frac{x_0}{\sigma_0^2} + \frac{y}{\Delta^2}\right)\sigma'^2 = x_0 \underbrace{\left(\frac{\Delta^2}{\sigma_0^2+\Delta^2}\right)}_{\equiv K_2} + y \underbrace{\left(\frac{\sigma_0^2}{\sigma_0^2+\Delta^2}\right)}_{K_1}$, using $\sigma'^2 = \frac{\sigma_0^2\Delta^2}{\sigma_0^2+\Delta^2}$

Note $K_2 = 1 - K_1 \Rightarrow x' = x_0 + K_1(y-x)$

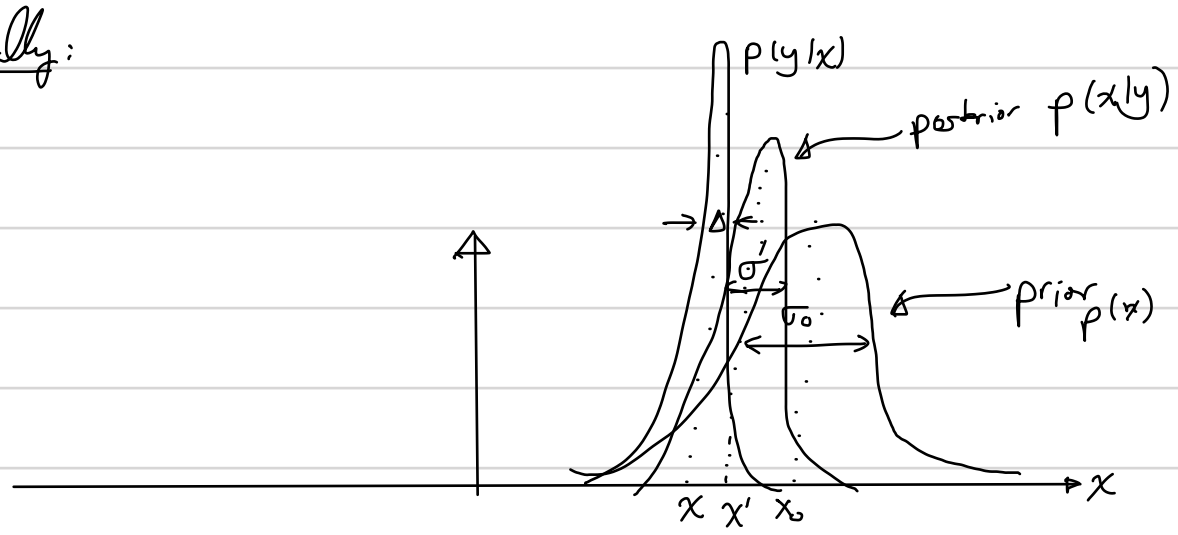
Thus, $p(y|x)\,p(x) = N e^{-\frac{(x-x')^2}{2\sigma'^2}}$ where N is all the factors that are independent of x

We can thus read off the normalization by eye without doing any integrals

$$\Rightarrow \boxed{p(x|y) = \frac{1}{\sqrt{2\pi\sigma'^2}}\, e^{-\frac{(x-x')^2}{2\sigma'^2}}} \quad \text{where } x' = x_0 + K_1(y-x),\ \sigma'^2 = K_2\sigma_0^2$$

$$K_1 = \frac{\sigma_0^2}{\sigma_0^2+\Delta^2},\quad K_2 = \frac{\Delta^2}{\sigma_0^2+\Delta^2}$$

Graphically:



After the measurement, our posterior probability distribution gets more refined (narrower) and closer to the true position x. As the resolution improves $\Delta \to 0$ $K_1 \to 1$ and $K_2 \to 0$, $\sigma' \to 0$

$\Rightarrow$ Posterior mean: $x' = y$, the measurement outcome with zero uncertainty.

Formally, the posterior probability distribution becomes a Dirac delta function

$p(x|y) \to \delta(x-y)$: If we measure y the particle is there.

(b) POVM for spin projection

We consider a spin J, with eigenvector of $\hat{J}_z$, $\hat{J}_z|M\rangle = M|M\rangle$. We define "Kraus Operators"

$$\hat{A}_\mu \equiv \sum_M \frac{1}{(2\pi\Delta^2)^{1/4}} e^{-\frac{(\mu-M)^2}{4\Delta^2}} |M\rangle\langle M| \quad \text{where } \mu \text{ is a real \#} \quad -\infty \le \mu \le \infty$$

Consider $\{ \hat{E}_\mu = \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{A}_\mu^2 = \sum_M \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(\mu-M)^2}{2\Delta^2}} |M\rangle\langle M| \text{ (function of an operator)} \}$

(since Hermitian)

- $\hat{E}_\mu \ge 0$: This is clear because $\frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{(\mu-M)^2}{2\Delta^2}} > 0$ are the eigenvalues of $\hat{E}_\mu$

- $\int_{-\infty}^{\infty} d\mu\, \hat{E}_\mu = \sum_M \underbrace{\int_{-\infty}^{\infty} d\mu \frac{e^{-\frac{(\mu-M)^2}{2\Delta^2}}}{\sqrt{2\pi\Delta^2}}}_{=1 \text{ (normalized Gaussian)}} |M\rangle\langle M| = \sum_M |M\rangle\langle M| = \hat{\mathbb{1}}$: resolution of the identity

$\Rightarrow \{\hat{E}_\mu\}$ forms a POVM

(c) In the limit of perfect resolution, $\Delta \to 0$, $\frac{e^{-\frac{(\mu-M)^2}{2\Delta^2}}}{\sqrt{2\pi\Delta^2}} \Rightarrow \delta(\mu - M)$ Dirac Delta

$$\hat{E}_\mu = \sum_M \delta(\mu - M)|M\rangle\langle M| = \sum_M \delta(\mu - M)|\mu\rangle\langle\mu| \quad (\text{Only non-zero when } M = \mu)$$

$\Rightarrow$ POVM become a projective measurement, as expected

(d) Given state prior to measurement: $|\psi_<\rangle = \sum_M \underbrace{\frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{(M-M_0)^2}{4\sigma_0^2}}}_{C_M} |M\rangle$

(i) $\langle \hat{J}_z \rangle_< = \langle\psi_<|\hat{J}_z|\psi_<\rangle = \sum_M M|C_M|^2 = \bar{M} = M_0$

$(\Delta J_z)_<$ is the rms of the probability distribution $|C_M|^2 = \sigma_0$

(ii) The probability of finding μ, $p(\mu)$, for measure outcome corresponding to $\hat{E}_\mu$

$$p(\mu) = \langle\psi_<|\hat{E}(\mu)|\psi_<\rangle = \sum_M \frac{1}{\pi\sigma_0\Delta} e^{-\frac{(\mu-M)^2}{2\Delta^2}} e^{-\frac{(M-M_0)^2}{2\sigma_0^2}}$$

This is exactly the conditional probability distribution we found for classical Gaussians in part (a).

(c) Conditioned on find μ, the post-measurement state follows from the generalization of the projection postulate:

$$|\psi_>\rangle = \frac{\hat{A}_\mu |\psi_<\rangle}{\| \hat{A}_\mu |\psi_<\rangle \|}$$

Aside: $\hat{A}_\mu |\psi_<\rangle = \sum_M \frac{1}{\sqrt{2\pi}\sigma_0 \Delta} e^{-\frac{(\mu - M)^2}{4\Delta^2}} e^{-\frac{(M-M_0)^2}{4\sigma_0^2}} |M\rangle = \sum_M \tilde{C}_M |M\rangle$

where $|\tilde{C}_M|^2 = \frac{1}{2\pi\sigma_0\Delta} e^{-\frac{(\mu-M)^2}{2\Delta^2}} e^{-\frac{(M-M_0)^2}{2\sigma_0^2}} = p(\mu|M)\, p(M)$, just as in part (a)

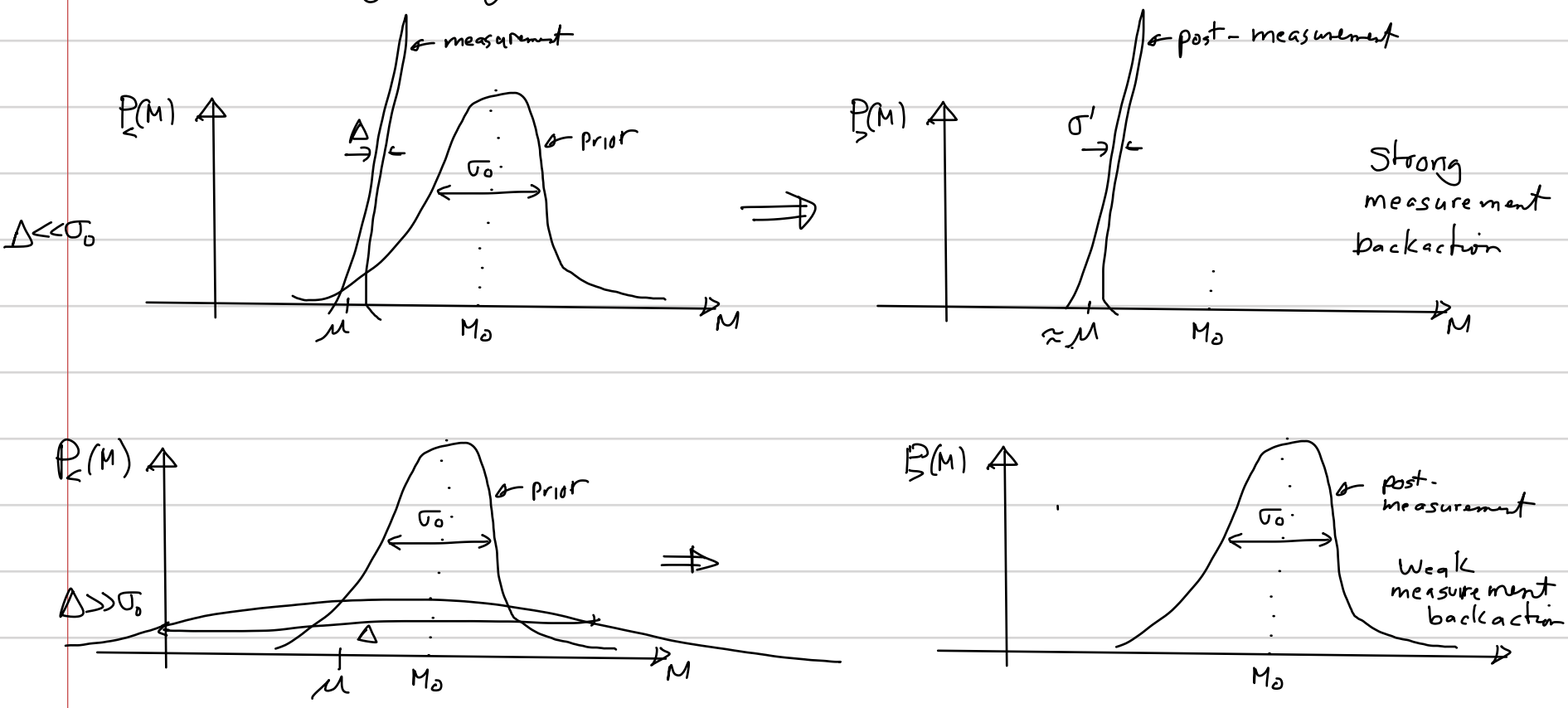
$\Rightarrow |\tilde{C}_M|^2 = N e^{-\frac{(M-M')^2}{2\sigma'^2}}$ where N is a constant that is reset by the normalizing denominator,

and $\boxed{\langle \hat{J}_z \rangle_> = M' = M_0 + \frac{\Delta^2}{\sigma_0^2 + \Delta^2}(\mu - M_0), \quad (\Delta J_z)_> = \sigma_0 \frac{\Delta}{\sqrt{\sigma_0^2 + \Delta^2}}}$

This is exactly as in part (a). The measurement "backaction" rule looks just like Bayesian updating. Of course, unlike in the classical case, the measurement backaction also affects our uncertainty about noncommuting observables!

(i) In the limit $\Delta \ll \sigma_0$ we approach a strong, projective measurement $M' \to \mu \quad (\Delta J_z)_> \to 0$

In the limit $\Delta \gg \sigma_0$ we have a very Weak measurement. In that case,

$\langle \hat{J}_z \rangle \approx M_0 \quad \Delta \hat{J}_z^2 \rangle \approx \sigma_0 \Rightarrow$ Measurement Backaction : negligible

This can be seen graphically