

Phy 521: Problem Set #4

Solutions

Problem 1: Some algebra with density operators

Given statistical mixture of spin- $\frac{1}{2}$ particles which is γ_3 in $|+z\rangle$ and γ_3 in $| -z\rangle$.

(a)

$$\Rightarrow \hat{\rho} = \frac{1}{3}|+z\rangle\langle +z| + \frac{2}{3}| -z\rangle\langle -z|$$

(Diagonal in this representation $\hat{\rho} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$)

Change basis $\hat{\rho} = \sum_{m_x=\pm} \sum_{m'_x=\pm} |m_x\rangle\langle m_x| \hat{\rho} |m'_x\rangle\langle m'_x|$

(Note we could use the matrices U from P.S. #1. Here, for variety, I will use Dirac notation)

$$\text{Recall } |\pm_x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$$

$$\begin{aligned} \Rightarrow \hat{\rho} = & \left(\frac{1}{3} |\langle +_x| +_z \rangle|^2 + \frac{2}{3} |\langle +_x| -_z \rangle|^2 \right) |+_x\rangle\langle +_x| \\ & + \left(\frac{1}{3} \langle +_x| +_z \rangle \langle +_z| -_x \rangle + \frac{2}{3} \langle +_x| -_z \rangle \langle -_z| -_x \rangle \right) |+_x\rangle\langle -_x| \\ & + \left(\frac{1}{3} \langle -_x| +_z \rangle \langle +_z| +_x \rangle + \frac{2}{3} \langle -_x| -_z \rangle \langle -_z| +_x \rangle \right) |-_x\rangle\langle +_x| \\ & + \left(\frac{1}{3} \langle -_x| +_z \rangle \langle +_z| -_x \rangle + \frac{2}{3} \langle -_x| -_z \rangle \langle -_z| -_x \rangle \right) |-_x\rangle\langle -_x| \end{aligned}$$

$$\Rightarrow \begin{aligned} \hat{\rho} = & \frac{1}{2} |+_x\rangle\langle +_x| - \frac{1}{6} |+_x\rangle\langle -_x| - \frac{1}{6} |-_x\rangle\langle +_x| \\ & + \frac{1}{2} |-_x\rangle\langle -_x| \end{aligned}$$

(Not diagonal in this representation, but still mixed)

$$\hat{\rho} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$(6) \quad \langle \hat{S}_y \rangle = \text{Tr}(\hat{\rho} \hat{S}_y)$$

There are many ways to approach this problem.
Here I will use the matrix representation in
the $|+\rangle_z$ basis.

$$\hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \hat{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \langle \hat{S}_x \rangle = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \right)$$

$$\Rightarrow \boxed{\langle \hat{S}_x \rangle = 0}$$

$$\langle \hat{S}_y \rangle = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix} \right)$$

$$\Rightarrow \boxed{\langle \hat{S}_y \rangle = 0}$$

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = \frac{\hbar}{2} \text{Tr} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \right)$$

$$= +\frac{\hbar}{2} \left(\frac{1}{2} - \frac{1}{2} \right) \Rightarrow \boxed{\langle \hat{S}_z \rangle = -\frac{\hbar}{2}}$$

For the completely mixed state $\hat{\rho} = \frac{1}{2} (|+\rangle_z \langle +|_z + |-\rangle_z \langle -|_z)$
 $= \frac{1}{2} \hat{I}$

$$\Rightarrow \boxed{\langle \hat{S}_y \rangle = \frac{1}{2} \text{Tr}(\hat{S}_y) = 0} \quad \text{for } x, y \text{ and } z$$

For a state which is a statistical mixture of $|+\rangle$ and $|-\rangle$ we will always have

$\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0$ ~~block~~ since $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ are zero for each member of the ensemble. However if $P_+ \neq P_-$ the spin ensemble is said to be "partially polarized" if $P_+ = P_- = \frac{1}{2}$ the sample is completely unpolarized.

$$(c) \quad \langle \hat{S}_n \rangle = \text{Tr}(\hat{\rho} \hat{S}_n) = \cos \theta \langle \hat{S}_z \rangle + \sin \theta (\cos \phi \langle \hat{S}_x \rangle + \sin \phi \langle \hat{S}_y \rangle)$$

For the $\frac{1}{3} - \frac{2}{3}$ mixture

$$\langle \hat{S}_n \rangle = -\frac{1}{6} \cos \theta$$

For the $\frac{1}{2} - \frac{1}{2}$ mixture

$$\langle \hat{S}_n \rangle = 0$$

The unpolarized sample has zero average projection of spin along any axis

Consider now a statistical mixture that is 50% spin-up along z and 50% spin-up along x .

$$\hat{\rho} = \frac{1}{2}|+x\rangle\langle+x| + \frac{1}{2}|+z\rangle\langle+z|$$

(d) The purity of the state $= \text{Tr}(\hat{\rho}^2)$

$$\hat{\rho}^2 = \frac{1}{4}|+x\rangle\langle+x| + \frac{1}{4}|+z\rangle\langle+z| + \frac{1}{4}|+x\rangle\langle+z| + \frac{1}{4}|+z\rangle\langle+x|$$

$$\text{Tr}(\hat{\rho}^2) = \underbrace{\frac{1}{4}\langle+x|x\rangle}_1 + \underbrace{\frac{1}{4}\langle+z|z\rangle}_1 + \underbrace{\frac{1}{4}|\langle+x|z\rangle|^2}_{\frac{1}{2}} + \underbrace{\frac{1}{4}|\langle+z|x\rangle|^2}_{\frac{1}{2}} \Rightarrow \text{Purity} = \frac{3}{4}$$

This state is not a completely mixed state. Such states have the minimum purity $= \frac{1}{d} = \frac{1}{2}$.

This state still has coherence. That is, there are bases in which it has nonzero off-diagonal matrix elements. E.g. in the $|+z\rangle$ basis $\hat{\rho} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

If is "partial polarized." This shows that 50-50 mixtures of spin- $\frac{1}{2}$ aren't completely mixed unless it is a mixture of two orthogonal states.

(e) We seek the eigenvectors and eigenvalues of $\hat{\rho}$. We can diagonalize the matrix in $|+z\rangle$ basis.

$$\text{Characteristic polynomial} = \det(\hat{\rho} - \lambda \hat{1}) = \det \begin{bmatrix} \frac{3}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} - \lambda \end{bmatrix} = \left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{16} = 0$$

$$\Rightarrow \lambda^2 - \lambda + \frac{1}{8} = 0 \Rightarrow \text{Eigenvalues } \lambda_{\pm} = \frac{1}{2} \pm \frac{\sqrt{1-4\lambda}}{2} = \frac{1}{2} \pm \frac{1}{2\sqrt{2}} = \frac{1}{2}(1 \pm \frac{1}{\sqrt{2}})$$

$$\text{Eigenvectors: } \begin{bmatrix} \frac{3}{4} - \lambda_{\pm} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} - \lambda_{\pm} \end{bmatrix} \begin{bmatrix} \alpha_{\pm} \\ \beta_{\pm} \end{bmatrix} = 0 \Rightarrow \left(\frac{3}{4} - \lambda_{\pm}\right)\alpha_{\pm} + \frac{\beta_{\pm}}{4} = 0 \Rightarrow \frac{\beta_{\pm}}{\alpha_{\pm}} = -3 + 4\lambda_{\pm} = -1 \pm \sqrt{2}$$

$$\begin{aligned} \text{Normalization } |\alpha_{\pm}|^2 + |\beta_{\pm}|^2 &= 1 = |\alpha_{\pm}|^2 \left(1 + \left|\frac{\beta_{\pm}}{\alpha_{\pm}}\right|^2\right) = |\alpha_{\pm}|^2 (4 \mp 2\sqrt{2}) = 1 \\ \Rightarrow |\alpha_{\pm}| &= \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} = \frac{\sqrt{2 \pm \sqrt{2}}}{2} = \frac{(\cos \pi/8)^{\pm 1}}{(\sin \pi/8)} \\ |\beta_{\pm}| &= \sqrt{\frac{2 \pm \sqrt{2}}{2}} (-1 \pm \sqrt{2}) = \sqrt{\frac{2 \pm \sqrt{2}}{2}} \end{aligned}$$

$$\Rightarrow |\lambda_{\pm}\rangle = \frac{\sqrt{2 \pm \sqrt{2}}}{2} |+z\rangle \pm \frac{\sqrt{2 \mp \sqrt{2}}}{2} |-z\rangle \Rightarrow \hat{\rho} = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-|$$

Note: We will see later a much simpler solution $\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{Q} \cdot \vec{\sigma})$
 $\vec{Q} = \frac{1}{2}\vec{e}_z + \frac{1}{2}\vec{e}_x = \text{"Bloch vector"}$