

# Physics 521

# Problem Set #5 Solutions

Prof 1

$$\hat{H} = \hbar\omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \hat{A} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{B} = b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle \quad \{ |u_i\rangle \text{ orthonormal}\}$$

(a) Measure <sup>energy of</sup> system at  $t=0$ :

Can find energy eigenvalues:

$$E_1 = \hbar\omega_0 \quad \text{with probability } P_1 = |\langle u_1 | \psi \rangle|^2 = \frac{1}{2}$$

$$E_2 = \hbar\omega_0 \quad \text{with probability } P_2 = |\langle u_2 | \psi \rangle|^2 + |\langle u_3 | \psi \rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(Since  $|u_2\rangle$  and  $|u_3\rangle$  are degenerate)

(b) Instead of  $\hat{H}$  we measure  $\hat{A}$ . We must find the eigenvectors/values of  $\hat{A}$ .

$\hat{A}$  is in "block diagonal" form

$$\hat{A} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly  $|u_1\rangle$  is an eigenvector of  $\hat{A}$  with eigenvalue  $a$ .  
 (Next page)

We must thus diagonalize  $\hat{A}$  in the subspace spanned by  $|u_2\rangle$  and  $|u_3\rangle$ :

Restricted  
to that  
subspace

$$\hat{A} = a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a \hat{\sigma}_x$$

where  $\hat{\sigma}_x$  is the familiar Pauli spin matrix whose in the basis where  $|+\rangle = |u_2\rangle$ ,  $|-\rangle = |u_3\rangle$ . We solved this problem in P.S. #1. The eigen-system is:

$$\lambda = a \quad |a\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle)$$

$$\lambda = -a \quad |-a\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(|u_2\rangle - |u_3\rangle)$$

$\hat{A}$  thus has pair of degenerated eigenvectors with eigenvalue "a", and a nondegenerate eigenvector  $|-a\rangle$

$$\lambda = a \quad |a^{(1)}\rangle = |u_2\rangle \quad |a^{(2)}\rangle = \frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle)$$

$$\lambda = -a \quad |-a\rangle = \frac{1}{\sqrt{2}}(|u_2\rangle - |u_3\rangle)$$

Suppose the state  $|\psi(0)\rangle$  enters the apparatus which "measures"  $\hat{A}$ . We can find:

- Eigenvalue  $-a$  with probability  $P_{-a} = |\langle -a | \psi(0) \rangle|^2$

$$\Rightarrow P_{-a} = \left| \underbrace{\langle u_2 |}_{\sqrt{2}} \langle u_3 | \right| |\psi(0)\rangle|^2 = \frac{1}{\sqrt{2}} \sqrt{\langle u_2 | \psi(0) \rangle^2 + \langle u_3 | \psi(0) \rangle^2}$$

$$= 0 \quad (\text{No component of } |-a\rangle \text{ in } |\psi(0)\rangle)$$

(If such a component had existed the state ~~at~~ immediately after the measurement would have been  $| -a \rangle$ )

- Eigenvalue  $+a$ : This is a degenerate subspace.

The probability of finding eigenvalue  $a$

$$P_a = \langle \psi(0) | \hat{P}_a | \psi(0) \rangle, \text{ where } \hat{P}_a = |a^{(1)}\rangle \langle a^{(1)}| + |a^{(2)}\rangle \langle a^{(2)}|$$

$$= |\langle a^{(1)} | \psi(0) \rangle|^2 + |\langle a^{(2)} | \psi(0) \rangle|^2$$

$$= |\langle u_1 | \psi(0) \rangle|^2 + \left| \left( \underbrace{\langle u_2 |}_{\sqrt{2}} + \langle u_3 | \right) \psi(0) \right|^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \quad (\text{as expected since } P_a + P_{-a} = 1)$$

Immediately after the measurement, according to Von Neumann projection postulate

$$|\psi_{\text{out}}\rangle = \frac{\hat{P}_a |\psi(0)\rangle}{\| \hat{P}_a |\psi(0)\rangle \|} = |\psi(0)\rangle \quad \begin{array}{l} \text{No change since} \\ |\psi(0)\rangle \text{ is an} \\ \text{eigenstate of } \hat{A} \end{array}$$

$$(c) |\psi(t)\rangle = U(t) |\psi(0)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar} |u_1\rangle + \frac{e^{-iE_2 t/\hbar}}{2} (|u_2\rangle + |u_3\rangle)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} |u_1\rangle + \frac{1}{2} e^{-i2\omega_0 t} (|u_2\rangle + |u_3\rangle)$$

$$(d) \langle \hat{A}(t) \rangle = \begin{bmatrix} e^{+i\omega_0 t} & e^{+i2\omega_0 t} & e^{+i4\omega_0 t} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega_0 t} \\ \frac{1}{\sqrt{2}} \\ \frac{-i2\omega_0 t}{2} \\ \frac{1}{2} \\ e^{-i4\omega_0 t} \end{bmatrix}$$

(Using representation  $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ )

$$\Rightarrow \langle \hat{A}(t) \rangle = \frac{a}{2} + \frac{a}{4} + \frac{a}{4} = a \quad \text{As expected since } |\psi(0)\rangle \text{ is eigenstate of}$$

$$\langle \hat{B}(t) \rangle = \begin{bmatrix} e^{i\omega_0 t} & e^{2i\omega_0 t} & e^{4i\omega_0 t} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} e^{-i\omega_0 t} \\ \frac{1}{\sqrt{2}} \\ \frac{-2i\omega_0 t}{2} \\ \frac{1}{2} \\ e^{-4i\omega_0 t} \end{bmatrix}$$

$$= \frac{b}{\sqrt{2}} \left( \frac{e^{-i\omega_0 t}}{2} + \frac{e^{i\omega_0 t}}{2} \right) + \frac{b}{4}$$

$$\langle \hat{B}(t) \rangle = \frac{b}{\sqrt{2}} \cos \omega_0 t + \frac{b}{4}$$

Note that  $\langle \hat{A} \rangle$  is independent of time while  $\langle \hat{B} \rangle$  is not. This follows directly from the fact that

$$[\hat{A}, \hat{A}] = 0 \quad \text{while} \quad [\hat{A}, \hat{B}] \neq 0$$

(c) Suppose we measure  $\hat{A}$  at time  $t$

Since  $[\hat{A}, \hat{A}] = 0$  the probabilities of measuring different values of  $\hat{A}$  do not change with time, thus, the results are the same as for part (b)

Suppose we measure  $\hat{B}$  at time  $t$ .

The eigenvectors/values can easily be found

Eigenvalue:  $b$  (doubly degenerate)

$$|b^{(1)}\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle) \quad |b^{(2)}\rangle = |u_3\rangle$$

②

Probability of finding "b" at time  $t$

$$P_b(t) = |\langle b^{(1)}|\psi(t)\rangle|^2 + |\langle b^{(2)}|\psi(t)\rangle|^2$$

$$= \left| \frac{e^{-ict}}{\sqrt{2}\sqrt{2}} + \frac{e^{-2ict}}{2\sqrt{2}} \right|^2 + \left| \frac{e^{-2ict}}{2} \right|^2$$

$$\boxed{P_b(t) = \frac{\sqrt{2}}{4} \cos(\omega t) + \frac{3}{8}}$$

Eigenvalue:  $-b$  (Nondegenerate)

$$P_{-b}(t) = |\langle -b|\psi(t)\rangle|^2 = \left| \left\langle u_1 \left| \frac{1}{\sqrt{2}}(u_1 + u_2) \right. \right\rangle \right|^2 = \left| \frac{e^{i\omega t}}{2} - \frac{e^{-i\omega t}}{2\sqrt{2}} \right|^2$$

$$\boxed{P_{-b}(t) = -\frac{\sqrt{2}}{4} \cos(\omega t) + \frac{3}{8}}$$

Note:

$$P_b(t) + P_{-b}(t) = 1, \quad \langle \hat{B} \rangle = b P_b(t) - b P_{-b}(t) = \frac{b}{\sqrt{2}} \cos \omega t + \frac{b}{4}$$

Problem Set #5

Problem 2

(a) Given time dependent Hamiltonian  $\hat{H}(t)$   
 the (local) eigenstates  $\{|u_n^{(t)}\rangle\}$   $\hat{H}(t)|u_n^{(t)}\rangle = E_n^{(t)}|u_n^{(t)}\rangle$

Note: Generally  $i\hbar \frac{\partial}{\partial t} |u_n^{(t)}\rangle \neq \hat{H}(t) |u_n^{(t)}\rangle$

We make the ansatz:

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-i\phi_n^{(t)}} |u_n^{(t)}\rangle$$

with  $\phi_n^{(t)} = \int_0^t E_n^{(t')} dt' / \hbar$

This reduces to the familiar form when  $\hat{H}$  is time independent. In that case  $c_n, E_n, |u_n\rangle$  are time independent  
 $\Rightarrow |\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |u_n\rangle$

Consider then the Schrödinger Equation for  $|\psi(t)\rangle$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

$$\Rightarrow \sum_n \left( i\hbar c_n e^{-i\phi_n^{(t)}} |u_n\rangle + \hbar \frac{d\phi_n}{dt} c_n e^{-i\phi_n^{(t)}} |u_n\rangle + c_n e^{-i\phi_n^{(t)}} i\hbar \frac{d}{dt} |u_n\rangle \right)$$

$$= \sum_n c_n e^{-i\phi_n^{(t)}} \hat{H}(t) |u_n^{(t)}\rangle = \sum_n c_n e^{-i\phi_n^{(t)}} E_n^{(t)} |u_n^{(t)}\rangle$$

where  $c_n = \frac{dc_n(t)}{dt}$  and  $\frac{d\phi_n^{(t)}}{dt} = E_n^{(t)} / \hbar$

Now take the inner product of both sides with  $\langle u_m^{(t)} |$  and use  $\langle u_m^{(t)} | u_n^{(t)} \rangle = \delta_{nm}$

$$\Rightarrow i\hbar \dot{c}_m e^{-i\phi_m^{(t)}} + E_m c_m e^{-i\phi_m^{(t)}} + i\hbar \sum_n c_n c_n^{(t)} \langle u_m^{(t)} | \frac{d}{dt} | u_n^{(t)} \rangle \\ = c_m e^{-i\phi_m^{(t)}} E_m$$

$$\Rightarrow \boxed{\dot{c}_m = - \sum_n c_n^{(t)} e^{-i(\phi_n^{(t)} - \phi_m^{(t)})} \langle u_m^{(t)} | \frac{d}{dt} | u_n^{(t)} \rangle}$$

(b) We must thus find  $\langle u_m^{(t)} | \frac{d}{dt} | u_n^{(t)} \rangle$

First consider the case  $n \neq m$

$$\hat{H}(t) |u_n^{(t)}\rangle = E_n^{(t)} |u_n^{(t)}\rangle$$

$$\Rightarrow \frac{\partial \hat{H}}{\partial t} |u_n^{(t)}\rangle + \hat{H}(t) \frac{\partial}{\partial t} |u_n^{(t)}\rangle = \frac{\partial E_n^{(t)}}{\partial t} |u_n^{(t)}\rangle + E_n^{(t)} \frac{\partial u_n^{(t)}}{\partial t}$$

Take inner product with  $\langle u_m^{(t)} |$   $m \neq n$

$$\Rightarrow \langle u_m^{(t)} | \frac{\partial \hat{H}}{\partial t} | u_n^{(t)} \rangle + \langle u_m^{(t)} | \hat{H}(t) \frac{\partial}{\partial t} | u_n^{(t)} \rangle \\ = \frac{\partial E}{\partial t} \langle u_m^{(t)} | u_n^{(t)} \rangle_0 + E_n^{(t)} \langle u_m^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle$$

Aside: Now use  $\langle u_m^{(t)} | \hat{H}(t) = E_m^{(t)} \langle u_m^{(t)} |$  (since  $\hat{H}(t)$  is Hermitian)

$$\Rightarrow \langle u_m^{(t)} | \frac{\partial \hat{H}}{\partial t} | u_n^{(t)} \rangle = (E_n^{(t)} - E_m^{(t)}) \langle u_m^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle$$

(Next Page)

Thus

$n \neq m$

$$\langle u_m^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle = \frac{\langle u_m^{(t)} | \frac{\partial \hat{H}}{\partial t} | u_n^{(t)} \rangle}{E_n^{(t)} - E_m^{(t)}}$$

For the case  $\hat{H}(t) = \hat{H}_0 + \frac{t}{T} \hat{H}_1$  (of  $\hat{H}$ ,

$$n \neq m \quad \boxed{\langle u_m^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle = \frac{\langle u_m^{(t)} | \hat{H}_1 | u_n^{(t)} \rangle}{i \omega_{nm}^{(t)}}} \quad \text{Where } \omega_{nm}^{(t)} = \frac{E_n^{(t)} - E_m^{(t)}}{T}$$

What about  $n = m$ ?

$$\text{First note: } \langle u_n^{(t)} | u_n^{(t)} \rangle = \int u_n^{(t)}(x)^* u_n^{(t)}(x) dx = 1$$

$$\Rightarrow \frac{d}{dt} \langle u_n^{(t)} | u_n^{(t)} \rangle = \int u_n^{(t)*} \frac{\partial}{\partial t} u_n^{(t)} + \int \left( \frac{\partial u_n^{(t)}}{\partial t} \right)^* u_n^{(t)} = 0$$

$$\Rightarrow \langle u_n^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle + \langle u_n^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle^* = 0$$

$\Rightarrow \langle u_n^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle$  is pure imaginary

Now, the phase of  $|u_n^{(t)}\rangle$  is arbitrary  $\Rightarrow$  we can always replace  $|u_n^{(t)}\rangle \Rightarrow e^{i\gamma^{(t)}} |u_n^{(t)}\rangle$  for some real  $\gamma^{(t)}$

$$\Rightarrow \langle u_n^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle \Rightarrow \langle \tilde{u}_n^{(t)} | \frac{\partial}{\partial t} | \tilde{u}_n^{(t)} \rangle + i\gamma^{(t)}$$

Thus through an appropriate choice of phase of the local eigenstate we can always set

$$\boxed{\langle u_n^{(t)} | \frac{\partial}{\partial t} | u_n^{(t)} \rangle = 0}$$

Thus, through an appropriate choice of phase

$$\dot{c}_m = - \sum_{n \neq m} c_n^{(t)} e^{-i \int_0^t \omega_{nm}^{(t')} dt'} \frac{\langle u_m^{(t)} | \hat{H}_1 | u_n^{(t)} \rangle}{\hbar \omega_{nm}^{(t)} T}$$

At this point we have made no approximation

Let us now assume we change  $\hat{H}$  very slowly,  
or adiabatically so that

$$\omega_{nm} T \gg 1 \Rightarrow \frac{1}{T} \ll \omega_{nm}^{(t)}$$

$\Rightarrow$  Rate of change of  $\hat{H}$  is much slower than  
the Bohr frequency associated with and

$$\text{transition } \omega_{nm} = \frac{E_n^{(t)} - E_m^{(t)}}{\hbar}$$

$$\Rightarrow \dot{c}_m \approx 0 \quad \forall m \Rightarrow \boxed{c_m^{(t)} = c_m^{(0)}}$$

After  
adiabatic  
change  $\Rightarrow \boxed{| \psi(t) \rangle = \sum_n c_n^{(0)} e^{-i \phi_n(t)} | u_n^{(t)} \rangle}$

The important point here is that the probability  
of occupying the  $n^{\text{th}}$  local eigenstate is independent  
of time

$$P_n(t) = \left| \langle u_n^{(t)} | \psi(t) \rangle \right|^2 = |c_n^{(0)} e^{-i \phi_n(t)}|^2$$

$$= |c_n^{(0)}|^2 = P_n(0)$$

(C) In contrast, suppose the Hamiltonian changes suddenly so that

$$T \ll \frac{\hbar}{\Delta E} \quad \text{where } \Delta E \text{ is the spread in energys at } t=0$$

Consider Schrödinger eqn:

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H(t) |\psi(t)\rangle$$

Integrate for time  $T$ . If  $T$  is small, then to lowest order

$$\begin{aligned} |\psi(T)\rangle &\approx |\psi(0)\rangle - \frac{i}{\hbar} \int_0^T dt' \hat{H}(t') |\psi(0)\rangle \\ &\approx |\psi(0)\rangle - \frac{iT}{\hbar} (\hat{H}_0 + \frac{1}{2}\hat{A}_1) |\psi(0)\rangle \end{aligned}$$

Now  $(\hat{H}_0 + \frac{1}{2}\hat{A}_1) |\psi(0)\rangle$  is no bigger than  $\Theta(\Delta E) |\psi(0)\rangle$

$$|\psi(T)\rangle \approx |\psi(0)\rangle - i \frac{T \Theta(\Delta E)}{\hbar} |\psi(0)\rangle$$

negligible

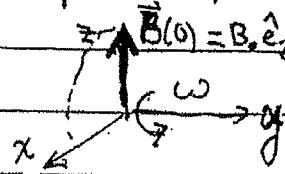
$\Rightarrow$  The wave function has not had time to change

$$\boxed{\begin{aligned} |\psi(T)\rangle &= |\psi(0)\rangle \\ &= \sum C_n^{(0)} |u_n^{(0)}\rangle \end{aligned}}$$

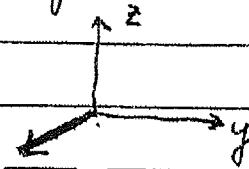
This is a form of the "time-energy" uncertainty principle. Given spread  $\Delta E$ , in time  $\Delta t \sim \hbar/\Delta E$  wavefunction does not change

### Problem 3

Spin  $\frac{1}{2}$  particle in a magnetic field



$$\vec{B}(0) = B_0 \hat{e}_z$$



Magnetic field rotates about -y-axis with frequency  $\omega$

$$\vec{B}(T) = B_0 \hat{e}_x$$

$$\vec{B}(t) = \begin{cases} +B_0 \hat{e}_z & t < 0 \\ B_0 \hat{e}_n(t) & 0 \leq t \leq T = \frac{\pi}{\omega} \\ -B_0 \hat{e}_x & t > T \end{cases} \quad \hat{e}_n(t) = t \cos \omega \hat{e}_z + \sin \omega t \hat{e}_x$$

The Hamiltonian for this system is time dependent

$$\hat{H}(t) = -\hat{\mu} \cdot \vec{B}(t) \quad \text{where } \hat{\mu} \text{ is the electron magnetic moment operator}$$

$$\Rightarrow \hat{H} = -\vec{\Omega}(t) \cdot \vec{s} \quad \hat{\mu} = \gamma \vec{s}, \quad \gamma = \text{gyromagnetic ratio}$$

$$\text{where } \vec{\Omega}(t) = \gamma \vec{B}(t) = \gamma B_0 \hat{e}_n(t)$$

Recall from problem 1 of P.S #4, when the field is static the spin precesses about  $-\hat{e}_n$  with frequency  $\Omega_n = \gamma B_0$  (Larmor frequency)



(a) For any time  $t$ , we have the instantaneous eigenstates and eigenvalues:

$$\hat{H}(t) |S_{n(t)}\rangle = E_{n(t)} |S_{n(t)}\rangle$$

$$-\Omega_0 (\vec{S} \cdot \vec{e}_n(t)) |S_{n(t)}\rangle = -S \hbar \Omega_0 |S_{n(t)}\rangle$$

$$\Rightarrow E_s(t) = -\frac{S \hbar \Omega_0}{2}, \quad S = \pm \frac{1}{2}, \quad \Omega_0 = \gamma B_0$$

$|S_{n(t)}\rangle$  are the eigenstates of  $(\vec{S} \cdot \vec{e}_n(t)) = \hat{S}_{n(t)}$   
(see Problem 1, P.S. #3)

At  $t=0$ , the electron is in its ground state

$$\Rightarrow |\psi(0)\rangle = |+_z\rangle = |+_z\rangle$$

$$\hat{S}_z |+_z\rangle = +\frac{\hbar}{2} |+_z\rangle \quad E_+ = -\frac{\hbar \Omega_0}{2}$$

According to the adiabatic theorem (see Problem 1, P.S. #5), if the change in  $\hat{H}(t)$  is:

$$(i) \text{ Sudden: } |\psi(T)\rangle = |\psi(0)\rangle = |+_z\rangle$$

$$(ii) \text{ Adiabatic: } |\psi(T)\rangle = e^{-i\phi_+(T)} |+_z\rangle$$

$$\text{where } \phi_+(T) = \int_0^T \frac{E_+(t)}{\hbar} dt' = -\frac{\Omega_0 T}{2}$$

$$|+_z\rangle = |+_x\rangle$$

$$|\psi(T)\rangle = e^{i\phi_+(T)/2} |+_x\rangle$$

(a) Continued:

For  $t > T$  the Hamiltonian is static,

$$t > T \quad \hat{H} = -\Omega_0 \hat{S}_x$$

with Eigenstates  $|+x\rangle$ , Eigenvalues  $\pm \frac{\hbar \Omega_0}{2}$

Thus for  $t > T$   $|\psi(+)\rangle = U(t, T) |\psi(T)\rangle$

$$U(t, T) = e^{-i \frac{\Omega_0}{2}(t-T)} = e^{+i \frac{\Omega_0}{2}(t-T)} |+x\rangle \langle +x| \\ + e^{-i \frac{\Omega_0}{2}(t-T)} |-x\rangle \langle -x|$$

$\Rightarrow$  If the change is sudden

$$t=0 \quad |\psi(0)\rangle = |+z\rangle$$

$$t=T \quad |\psi(T)\rangle = |+z\rangle = \frac{1}{\sqrt{2}} (|+x\rangle + |-x\rangle) \quad (\text{See problem P.S. #})$$

$$\Rightarrow t > T \quad |\psi(+)\rangle = \frac{1}{\sqrt{2}} \left( e^{+i \frac{\Omega_0}{2}(t-T)} |+x\rangle + e^{-i \frac{\Omega_0}{2}(t-T)} |-x\rangle \right)$$

If the change is adiabatic

$$t=0 \quad |\psi(0)\rangle = |+z\rangle$$

$$t=T \quad |\psi(T)\rangle = e^{+i \frac{\Omega_0 T}{2}} |+x\rangle$$

$$t > T \quad |\psi(+)\rangle = e^{i \frac{\Omega_0}{2}(t-T)} e^{i \frac{\Omega_0 T}{2}} |+x\rangle$$

$$\Rightarrow |\psi(+)\rangle = e^{i \frac{\Omega_0 t}{2}} |+x\rangle$$

(a) Continued

Condition for adiabatic change vs. sudden change

According to the adiabatic theorem, we require must consider the Bohr frequencies. In this case, with only 2 states there is only one Bohr frequency

$$\omega_{\text{Bohr}} = \frac{E_- - E_+}{\hbar} = \frac{\frac{h\nu_0}{2} - \left(\frac{h\nu_0}{2}\right)}{\hbar} = \nu_0 \quad (\text{The Larmor frequency})$$

Adiabatic: Require rate of change of  $H \ll \omega_{\text{Bohr}}$

$$\Rightarrow \boxed{\omega \ll \nu_0} \quad \text{or} \quad \boxed{T \gg \frac{1}{\nu_0}}$$

Sudden: Rate of change of  $H \gg \omega_{\text{Bohr}}$

$$\Rightarrow \boxed{\omega \gg \nu_0} \quad \text{or} \quad \boxed{T \ll \frac{1}{\nu_0}}$$

(b) Probability of finding electron in original state:

$$P(t) = K |\psi(0)| \langle \psi(t) \rangle^2 = \left| \langle +_z | \psi(t) \rangle \right|^2$$

• Sudden:  $|\psi(t)\rangle = \frac{1}{\sqrt{2}} (e^{i\nu_0 t/2} |+_x\rangle + e^{-i\nu_0 t/2} |-_x\rangle)$   
(setting  $\hbar, \nu_0 T \rightarrow 0$  in sudden approx)

$$\Rightarrow P(t) = \frac{1}{2} \left| e^{i\nu_0 t/2} \langle +_z | +_x \rangle + e^{-i\nu_0 t/2} \langle +_z | -_x \rangle \right|^2$$

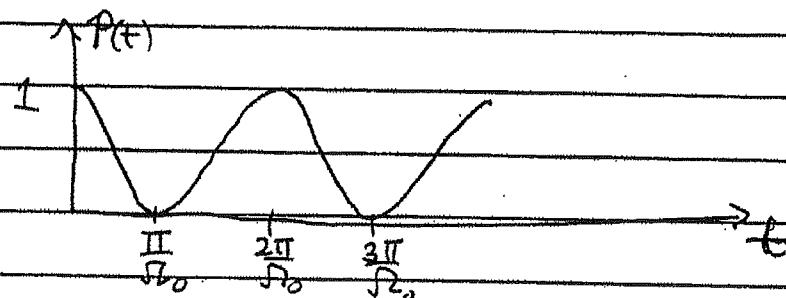
(Aside:  $|+_x\rangle = \frac{1}{\sqrt{2}} (|+_z\rangle \pm |-_z\rangle) \Rightarrow \langle +_z | +_x \rangle = \frac{1}{\sqrt{2}}$ )

(Next Page)

(i)  $\Rightarrow$  In the "sudden approximation", the probability to be found in  $|+\rangle_z$  for  $t > T$  is

$$P(t) = \frac{1}{4} \left| e^{\frac{i\Omega_0 t}{2}} + e^{-i\frac{\Omega_0 t}{2}} \right|^2 = \cos^2\left(\frac{\Omega_0 t}{2}\right)$$

$$\Rightarrow P(t) = \frac{1}{2} (1 + \cos(\Omega_0 t)) \quad \text{Sudden}$$



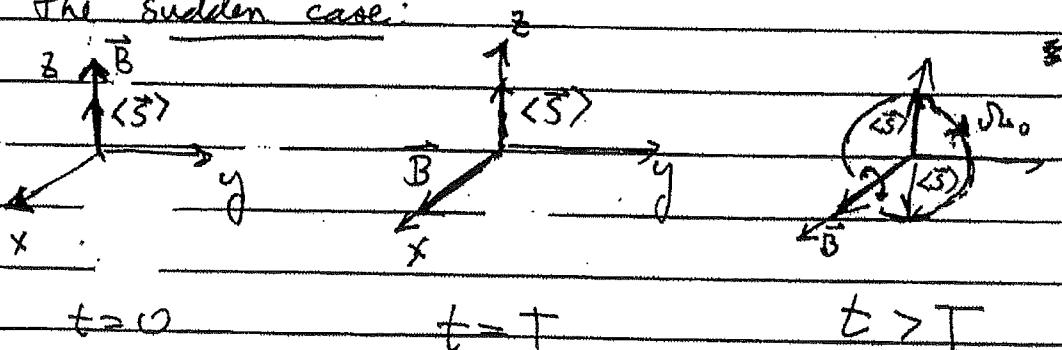
(ii) In the adiabatic case

$$P(t) = \left| e^{-i\frac{\Omega_0 t}{2}} \langle + | + \rangle \right|^2 = |\langle +_z | +_x \rangle|^2$$

$$\Rightarrow P(t) = \frac{1}{2} \quad \text{Adiabatic.}$$

We can understand this semi-classically.

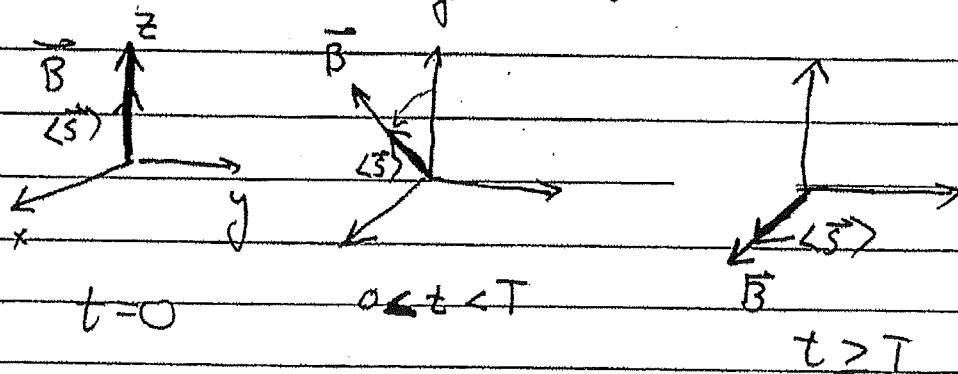
(i) In the sudden case:



At time  $t = T$  the system is no longer an eigenstate and so the state will evolve non-trivially with time. In a semi-classical picture the spin vector precesses (Next Page)

about  $-\vec{B}$  (i.e. the  $\hat{\vec{c}}_x$  axis) with frequency  $\omega_z$ , and thus the probability to be in  $|+z\rangle$  oscillates with frequency  $\omega_z$ .

(ii) In the adiabatic case the spin "adiabatically-follows" the magnetic field, and always remains in the instantaneous ground state



Thus the spin is always in a stationary state and probabilities of measurements are time independent. In this case the spin is adiabatically evolved into  $|+x\rangle$  which has a 50% probability of being measured in  $|+z\rangle$

### (c) The Heisenberg picture

- States independent of time  $|\psi\rangle_H = |\psi(0)\rangle_S = |+z\rangle$

- Operators evolve in time according to the equation of motion

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar} [\hat{A}, \hat{H}(t)]$$

Consider the  $i^{\text{th}}$  component of the spin operator

$$\frac{d}{dt} \hat{S}_i = \frac{1}{i\hbar} [\hat{S}_i, H(t)] = -\frac{1}{i\hbar} [\hat{S}_i, \vec{\Omega}(t) \cdot \hat{S}]$$

$$= - \sum_j \Omega_{ij}(t) [\hat{S}_i, \hat{S}_j] = - \sum_k g_{ik} \Omega_k(t) \hat{S}_k$$

Or in vector form:

$$\forall t \leq T \quad \frac{d}{dt} \hat{\vec{S}} = - \vec{\Omega}(t) \times \hat{\vec{S}}, \quad \text{where } \vec{\Omega}(t) = \Omega_0 \vec{e}_n(t), \quad \Omega_0 = \gamma B_0$$

To solve the equation we go to the rotating frame, rotating about  $y$ -axis with frequency  $\omega$ .

In this frame the  $\vec{B}$  field is static, but there is a "pseudo force". Denoting the vectors in the frame with primes:

$$\frac{d}{dt} \hat{\vec{S}}' = - \vec{\Omega}' \times \hat{\vec{S}}' - \underbrace{\vec{\omega} \times \hat{\vec{S}}'}_{\text{pseudo-force}}$$

$$\text{where } \vec{\Omega}' = \Omega_0 \vec{e}_{z'}, \quad \vec{\omega} = \omega \vec{e}_{y'} = \omega \vec{e}_y$$

(Note  $\vec{e}_{z'} = \vec{e}_z(t)$ :  $\vec{B}$  field points along the instantaneous  $z$ -direction)

In component form:

$$\frac{d\hat{S}'_x}{dt} = \Omega_0 \hat{S}'_y, \quad \frac{d\hat{S}'_y}{dt} = -\Omega_0 \hat{S}'_x, \quad \frac{d\hat{S}'_z}{dt} = \omega \hat{S}'_x - \omega \hat{S}'_z$$

Let's look at the adiabatic case first:

Since  $\omega \ll \Omega_0$ , we can neglect its contribution to  $\frac{d\hat{S}_x}{dt}$ .

$$\Rightarrow \frac{d\hat{S}_x}{dt} \approx \Omega_0 \hat{S}_y, \quad \frac{d\hat{S}_y}{dt} = -\Omega_0 \hat{S}_x$$

$$\Rightarrow \hat{S}_x(t) = \hat{S}_x(0) \cos(\Omega_0 t) + \hat{S}_y(0) \sin(\Omega_0 t)$$

$$\hat{S}_y(t) = \hat{S}_y(0) \cos(\Omega_0 t) - \hat{S}_x(0) \sin(\Omega_0 t)$$

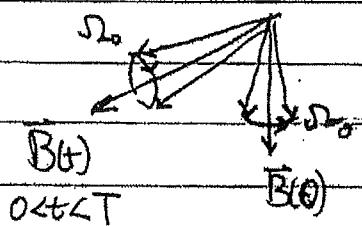
$$\Rightarrow \frac{d\hat{S}_z}{dt} = \omega \hat{S}_x \Rightarrow \hat{S}_z(t) = \left\{ \omega \int_0^t \hat{S}_x(t') dt' + \hat{S}_z(0) \right\}$$

$$\Rightarrow \hat{S}_z(t) = \hat{S}_z(0) + \left[ \frac{\omega}{\Omega_0} \left( \hat{S}_x(0) \sin \Omega_0 t - \hat{S}_y(0) \cos \Omega_0 t \right) \right]$$

neglect  $\omega$  in adiabatic approximation

$$\Rightarrow \hat{S}_z(t) = \hat{S}_z(0)$$

$\Rightarrow$  In the rotating frame, in the adiabatic case, the spin precesses rapidly around the local  $\vec{B}$ . Oscillating many times before  $\vec{B}$  changes appreciably.



(Classical picture of  
adiabatic following)

Finally we transform back to the lab frame

$$\Rightarrow \hat{S}_x(t) = \hat{S}_{x\sigma}(t) \cos\omega t + \hat{S}_{z\sigma}(t) \sin\omega t$$

$$\Rightarrow \hat{S}_x(T = \frac{\pi}{2\omega}) = \hat{S}_{z\sigma}(T) \approx \hat{S}_z(0)$$

$$\Rightarrow \text{In adiabatic case (a)} \quad \boxed{\hat{S}_x(T) \approx \hat{S}_z(0)}$$

In the sudden case  $T \ll \frac{1}{\Omega_{10}}$

$$\Rightarrow \frac{d\hat{S}}{dt} \approx 0 \Rightarrow \hat{S}(t) \approx \hat{S}(0)$$

$$\Rightarrow \boxed{\hat{S}_x(T) \approx \hat{S}_x(0)} \text{ Sudden}$$

(d) Using the Schrödinger picture solution in (a)

$$|\psi(T)\rangle = \begin{cases} |+_z\rangle: \text{ Sudden} \\ (e^{i\Omega_0 t/2} |+_x\rangle): \text{ Adiabatic} \end{cases}$$

$$\Rightarrow \langle \hat{S}_x(T) \rangle = \begin{cases} \langle +_z | \hat{S}_x | +_z \rangle = 0 & \text{Sudden} \\ \langle +_x | \hat{S}_x | +_x \rangle = \frac{\hbar}{2} & \text{Adiabatic} \end{cases}$$

a Using the Heisenberg picture of part (c)

$$|\psi\rangle = |+_z\rangle \text{ for all times where } \hat{S}_z(0)|+_z\rangle = \frac{\hbar}{2}|+_z\rangle$$

$$\Rightarrow \langle \hat{S}_x(T) \rangle = \begin{cases} \langle \psi | \hat{S}_x(T) | \psi \rangle = \langle +_z | \hat{S}_x(0) | +_z \rangle = 0 & \text{Sudden} \\ \langle \psi | \hat{S}_x(T) | \psi \rangle = \langle +_z | \hat{S}_x(0) | +_z \rangle = \frac{\hbar}{2} & \text{adiabatic} \end{cases}$$

