

Physics 521

Problem Set #6

Solutions

Problem 4: Spreading in the Heisenberg Picture:

(a) Heisenberg's Eqns of motion for an arbitrary operator $\frac{d\hat{A}}{dt} = \frac{1}{i\hbar} [\hat{A}, \hat{H}] \Rightarrow \frac{d\langle \hat{A} \rangle_t}{dt} = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle_t$

Free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\begin{aligned} \frac{d\langle \hat{x} \rangle_t}{dt} &= \frac{1}{i\hbar} \langle [\hat{x}, \hat{H}] \rangle_t = \frac{1}{2m} \frac{1}{i\hbar} \langle [\hat{x}, \hat{p}^2] \rangle_t \\ &= \frac{1}{2m} \frac{1}{i\hbar} \langle \hat{p} [\hat{x}, \hat{p}] + [\hat{x}, \hat{p}] \hat{p} \rangle_t = \frac{1}{2m} \frac{1}{i\hbar} \langle 2i\hbar \hat{p} \rangle_t \end{aligned}$$

$$\Rightarrow \boxed{\frac{d\langle \hat{x} \rangle_t}{dt} = \frac{\langle \hat{p} \rangle_t}{m}} \quad \text{Classical eqn of motion}$$

$$\boxed{\frac{d\langle \hat{p} \rangle_t}{dt} = \frac{1}{i\hbar} \langle [\hat{p}, \hat{H}] \rangle_t = 0} \quad \left(\begin{array}{l} \text{Aside} \\ \frac{d\langle \hat{p}^2 \rangle_t}{dt} = 0 \text{ for} \\ \text{some} \\ \text{reason} \end{array} \right)$$

$$\begin{aligned} \frac{d\langle \hat{x}^2 \rangle_t}{dt} &= \frac{1}{2m} \frac{1}{i\hbar} \langle [\hat{x}^2, \hat{p}^2] \rangle_t = \frac{1}{2m} \frac{1}{i\hbar} \langle \hat{x} [\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2] \hat{x} \rangle_t \\ &= \frac{1}{2m} \frac{1}{i\hbar} \langle \hat{x} (2i\hbar \hat{p}) + (2i\hbar \hat{p}) \hat{x} \rangle_t \end{aligned}$$

$$\Rightarrow \boxed{\frac{d\langle \hat{x}^2 \rangle_t}{dt} = \frac{1}{2m} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t}$$

$$\begin{aligned} \frac{d\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t}{dt} &= \frac{1}{i\hbar} \left(\langle [\hat{x} \hat{p}, \hat{H}] \rangle_t + \langle [\hat{p} \hat{x}, \hat{H}] \rangle_t \right) \\ &= \frac{1}{i\hbar} \left(\langle [\hat{x}, \hat{H}] \rangle_t \hat{p} + \hat{p} \langle [\hat{x}, \hat{H}] \rangle_t \right) \end{aligned}$$

$$\Rightarrow \boxed{\frac{d\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t}{dt} = \frac{2}{m} \langle \hat{p}^2 \rangle_t}$$

(b) These equations can now be easily integrated

$$\langle \hat{p} \rangle_t = \langle \hat{p} \rangle_0 \quad \langle \hat{p}^2 \rangle_t = \langle \hat{p}^2 \rangle_0$$

$$\Rightarrow \frac{d}{dt} \langle \hat{x} \rangle_t = \frac{\langle \hat{p} \rangle_0}{m} \Rightarrow \boxed{\langle \hat{x} \rangle_t = \langle \hat{x} \rangle_0 + \frac{\langle \hat{p} \rangle_0}{m} t}$$

$$\frac{d}{dt} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t = \frac{2}{m} \langle \hat{p}^2 \rangle_0 \Rightarrow \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t = \frac{2t}{m} \langle \hat{p}^2 \rangle_0 + \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0$$

$$\Rightarrow \frac{d}{dt} \langle \hat{x}^2 \rangle_t = \frac{1}{m} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_t = \frac{2t}{m^2} \langle \hat{p}^2 \rangle_0 + \frac{1}{m} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0$$

$$\Rightarrow \boxed{\langle \hat{x}^2 \rangle_t = \langle \hat{x}^2 \rangle_0 + \frac{\langle \hat{p}^2 \rangle_0}{m^2} t^2 + \frac{1}{m} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 t}$$

$$(c) \langle (\Delta \hat{x})^2 \rangle_t = \langle \hat{x}^2 \rangle_t - \langle \hat{x} \rangle_t^2$$

$$= \langle \hat{x}^2 \rangle_0 + \frac{\langle \hat{p}^2 \rangle_0}{m^2} t^2 + \frac{1}{m} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 t$$

$$- \left(\langle \hat{x} \rangle_0 + \frac{\langle \hat{p} \rangle_0}{m} t \right)^2$$

$$= \left(\langle \hat{x}^2 \rangle_0 - \langle \hat{x} \rangle_0^2 \right) + \frac{\left(\langle \hat{p}^2 \rangle_0 - \langle \hat{p} \rangle_0^2 \right)}{m^2} t^2$$

$$+ \frac{2t}{m} \left(\frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 - \langle \hat{x} \rangle_0 \langle \hat{p} \rangle_0 \right)$$

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Thus,

$$\langle (\Delta \hat{x})^2 \rangle_t = \langle (\Delta \hat{x})^2 \rangle_0 + \frac{\langle (\Delta \hat{p})^2 \rangle_0}{m^2} t^2 + \frac{2t}{m} \left(\frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 - \langle \hat{x} \rangle_0 \langle \hat{p} \rangle_0 \right)$$

Let us consider the wave packet given in Problem 3. In the Heisenberg picture, the state is the one given by the Schrödinger picture at $t=0$

$$\Rightarrow \psi_{\text{H}}(x) = \psi_{\text{S}}(x, 0) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{x^2}{4\sigma_0^2}} e^{ik_0 x}$$

$$\Rightarrow \langle (\Delta \hat{x})^2 \rangle_0 = \sigma_0^2 \quad \langle (\Delta \hat{p})^2 \rangle_0 = \left(\frac{\hbar}{2\sigma_0} \right)^2$$

$$\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 = \langle \hat{x} \hat{p} + [\hat{p}, \hat{x}] \rangle_0 = -i\hbar + 2\langle \hat{x} \hat{p} \rangle_0$$

$$\langle \hat{x} \hat{p} \rangle_0 = -i\hbar \int_{-\infty}^{\infty} \psi(x, 0) x \frac{\partial}{\partial x} \psi(x, 0) = -i\hbar \int_{-\infty}^{\infty} \left(\frac{-x^2}{2\sigma_0^2} \right) |\psi(x, 0)|^2 dx$$

$$= i\hbar \frac{\langle \hat{x}^2 \rangle_0}{\sigma_0^2} = i\hbar$$

$$\Rightarrow \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 = 0$$

$$\langle \hat{x} \rangle_0 \langle \hat{p} \rangle_0 = (0)(\hbar k_0) = 0$$

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Q Thus

$$\langle \hat{x}^2 \rangle_t = \sigma_0^2 + \left(\frac{\hbar}{2m\sigma_0} \right)^2 t^2 \quad \text{in agreement with Problem 3!}$$

There are a few comments that should be made:

$$\text{Let us suppose } \frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle_0 - \langle \hat{x} \rangle_0 \langle \hat{p} \rangle_0 = 0$$

(We'll come back to understand the physical meaning below)

$$\Rightarrow \langle (\Delta \hat{x})^2 \rangle_t = \langle (\Delta \hat{x})^2 \rangle_0 + \frac{\langle (\Delta \hat{p})^2 \rangle_0}{m^2} t^2$$

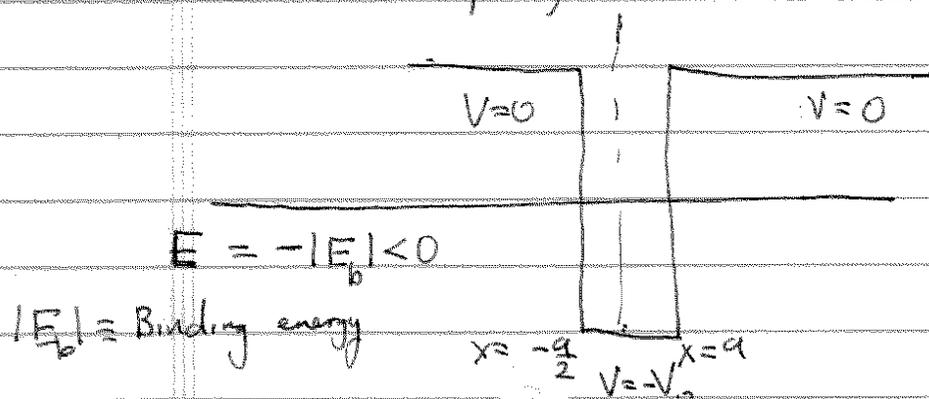
This is the standard expression for spreading of a wavepacket. Physically, the packet contains a spread in momenta. Thus different Fourier components move with different phase velocities and the packet spreads apart. The rate of spreading depends on Δp_0 . However, because of the uncertainty principle $\Delta p_0 \sim \frac{\hbar}{\Delta x_0}$. Thus, the more localized the packet is at $t=0$, the faster it spreads apart. We can characterize this by finding the ~~distance~~^{time} the packet must travel before ~~double~~ doubling its width.

$$\frac{\langle (\Delta \hat{p})^2 \rangle_0}{m^2} T_{\text{char}}^2 = \langle (\Delta \hat{x})^2 \rangle_0 = \frac{\hbar^2}{4m^2 \langle (\Delta \hat{x})^2 \rangle_0} T_{\text{char}}^2$$

$$\Rightarrow T_{\text{char}} = \frac{2m \langle (\Delta \hat{x})^2 \rangle_0}{\hbar}$$

Problem 2 The Delta function potential

- (a) Consider the finite square well of width, a , and depth, V_0 .



Note: Here $V=0$ outside the well and $V=-V_0$ inside the well

What ^{are} the bound states of this system in the limit $a \rightarrow 0$ $V_0 \rightarrow \infty$ with $aV_0 \rightarrow V_0$ (constant)?

In class we studied the finite well, whose bound states are the simultaneous solutions of

• even parity: $ka \tan\left(\frac{ka}{2}\right) = Ka$ and $(ka)^2 + (Ka)^2 = (k_0 a)^2$

• odd parity: $-ka \cot\left(\frac{ka}{2}\right) = Ka$ and "

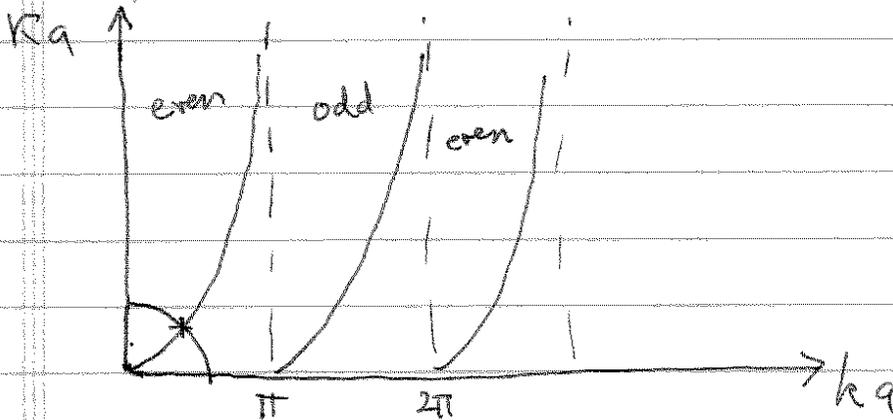
where $k \equiv \sqrt{\frac{2m}{\hbar^2}(E - V_{\text{inside}})} = \left(\sqrt{\frac{2m}{\hbar^2}(V_0 - |E_b|)} \right)$
 in this case

$K \equiv \sqrt{\frac{2m}{\hbar^2}(V_{\text{outside}} - E)} = \left(\sqrt{\frac{2m}{\hbar^2}|E_b|} \right)$

$k_0 \equiv \sqrt{\frac{2m}{\hbar^2}V_0}$

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Graphical Solution:



Limit $a \rightarrow 0$ $V_0 \rightarrow \infty$
 $V_0 a \rightarrow U_0$

$$k_0 a \propto \sqrt{V_0} a \rightarrow 0$$

$$\Rightarrow \text{Radius of circle} \rightarrow 0$$

$$(ka)^2 + (Ka)^2 = (k_0 a)^2 \rightarrow 0$$

\Rightarrow Only one bound state
 Even Parity

Limit

$$\left. \begin{aligned} ka &\rightarrow \sqrt{\frac{2m}{\hbar^2} V_0} a \\ Ka &\rightarrow \sqrt{\frac{2m}{\hbar^2} |E_b|} a \end{aligned} \right\} \Rightarrow ka \tan\left(\frac{ka}{2}\right) = Ka$$

$$\approx \frac{(ka)^2}{2} = Ka \Rightarrow \frac{2m V_0 a^2}{\hbar^2} = \sqrt{\frac{2m |E_b|}{\hbar^2}} a$$

$$\therefore |E_b| = \frac{m U_0^2}{2 \hbar^2} \quad \text{binding energy}$$

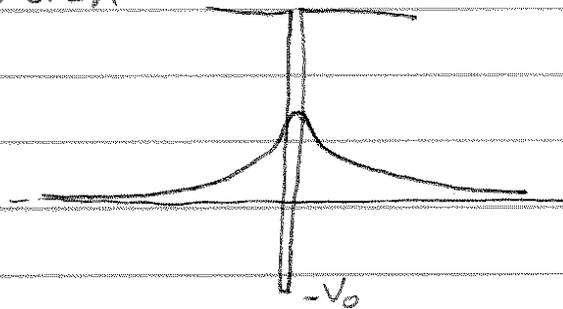
Wave function: Only "outside" the delta function

$$u(x) = A e^{-\gamma |x|} \quad \gamma = \kappa = \sqrt{\frac{2m}{\hbar^2} |E_b|}$$

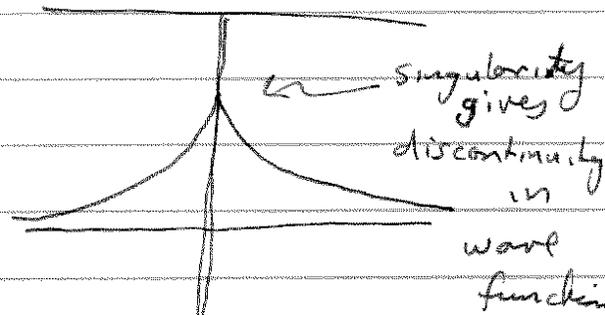
Normalization: $\int_{-\infty}^{\infty} |u(x)|^2 dx = 2A^2 \int_0^{\infty} e^{-2\gamma x} dx = 1 \Rightarrow \gamma = \frac{2m U_0}{\hbar^2}$

$$\Rightarrow A = \sqrt{\gamma} \quad \therefore u(x) = \sqrt{\gamma} e^{-\gamma |x|}$$

Sketch



"Skinny" finite well



"Delta-function"

(b) Solving using the boundary conditions.

We seek solution to

$$-\frac{\hbar^2}{2m} \frac{d^2 u(x)}{dx^2} - U_0 \delta(x) u(x) = -|E_b| u(x)$$

We have the solutions

$$x < 0 \quad u_{\leftarrow}(x) = A e^{\gamma x}$$

$$x > 0 \quad u_{\rightarrow}(x) = B e^{-\gamma x}$$

$$\gamma = \sqrt{\frac{2m}{\hbar^2} |E_b|}$$

Boundary Conditions

• $u_{\leftarrow}(0) = u_{\rightarrow}(0) \Rightarrow A = B \equiv u(0)$

• What about the derivative? From the sketch above, it is clear that the derivative is discontinuous due to the discontinuity. Formally, let us integrate across the delta functions:

$$\lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \left(-\frac{\hbar^2}{2m} \frac{d^2 u}{dx^2} - U_0 \delta(x) u(x) \right) dx = -|E_b| \int_{-\epsilon}^{\epsilon} u(x) dx$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\frac{\partial u}{\partial x} \Big|_{-\epsilon} - \frac{\partial u}{\partial x} \Big|_{+\epsilon} \right] + \frac{2mU_0}{\hbar^2} u(0) = -|E_b| \lim_{\epsilon \rightarrow 0} u(0) \epsilon$$

mean value theorem

In the limit $\epsilon \rightarrow 0$ the right hand side goes to zero

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \left[\frac{\partial u}{\partial x} \Big|_{-\epsilon} - \frac{\partial u}{\partial x} \Big|_{+\epsilon} \right] = \frac{2mU_0}{\hbar^2} u(0)$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} [\gamma A e^{+\gamma \epsilon} + \gamma A e^{-\gamma \epsilon}] = 2\gamma A = \frac{2mU_0}{\hbar^2} A$$

$$\Rightarrow 2\gamma = 2\sqrt{\frac{2m}{\hbar^2} |E_b|} = \frac{2mU_0}{\hbar^2}$$

$$\Rightarrow \boxed{|E_b| = \frac{m}{2\hbar^2} U_0^2}$$

Binding energy as before

(c) Now we have two delta function potential wells

$$V(x) = -U_0 \delta(x-b) - U_0 \delta(x+b)$$

$x = -b$


$x = +b$


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In the limit $b \gg \frac{1}{\gamma}$, a particle bound in one potential will not "feel" the effect of the other



Thus there are two almost degenerate (for all intents and purpose) eigenfunction

$$u_R(x) = \sqrt{\gamma} e^{-\gamma(x-b)} \quad (\text{Particle localized at the right potential})$$

$$u_L(x) = \sqrt{\gamma} e^{-\gamma(x+b)} \quad (\text{Particle localized at the left potential})$$

$$\text{with } H u_R(x) \approx -|E_b| u_R(x)$$

$$H u_L(x) \approx -|E_b| u_L(x)$$

We can construct eigenstates of parity from the degenerate subspace

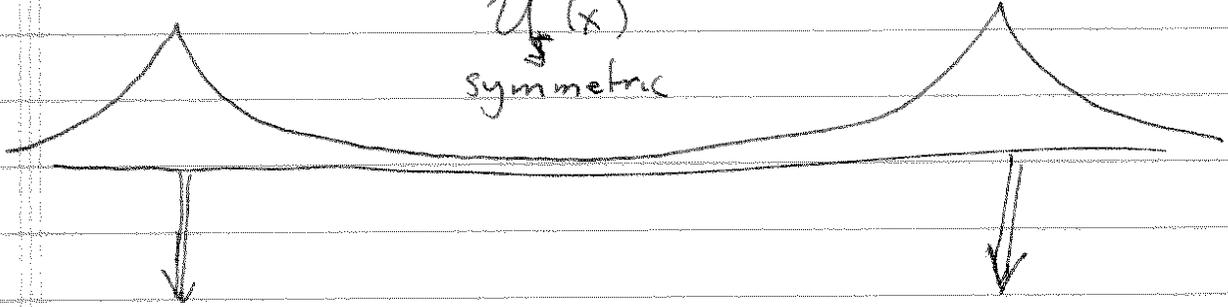
$$\text{Symmetric: } u_S(x) \equiv \frac{1}{\sqrt{2}} (u_R(x) + u_L(x)) = \frac{\sqrt{\gamma}}{\sqrt{2}} (e^{-\gamma|x-b|} + e^{-\gamma|x+b|})$$

$$\text{Antisymmetric: } u_A(x) \equiv \frac{1}{\sqrt{2}} (u_R(x) - u_L(x)) = \frac{\sqrt{\gamma}}{\sqrt{2}} (e^{-\gamma|x-b|} - e^{-\gamma|x+b|})$$

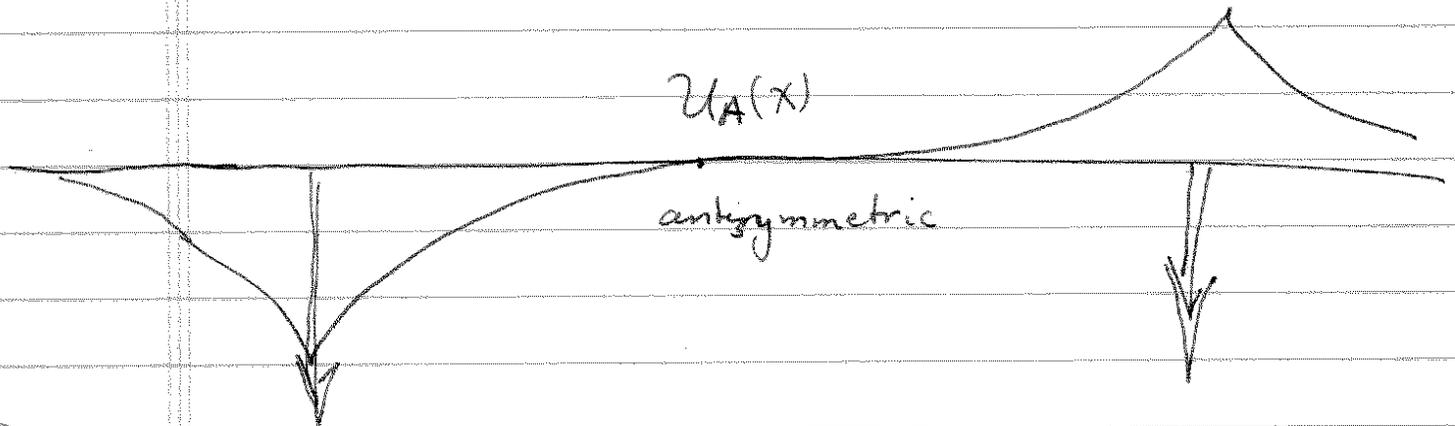
(Note the normalization neglects any overlap between these functions u_R and u_L)

Sketch

$\psi_s(x)$
symmetric



$\psi_A(x)$
antisymmetric



The symmetric state must be the true ground state since it has zero nodes. The antisymmetric state having one node has more curvature (inflection points) which corresponds to a higher energy.

- (d) If we take into account the small overlap of $\psi_R(x)$ and $\psi_L(x)$ then the previously degenerate $\psi_s(x)$ and $\psi_A(x)$ are split in energy $\Delta E = E_A - E_S$.
Suppose at $t=0$ the particle is localized around one of the delta functions:

$$\psi(x, 0) = \psi_R(x)$$

What is $\psi(x, t)$?

To find $\psi(x,t)$ we decompose $\psi(x,0)$ into the approximate eigenstates. Given the state $u_{S,A} \equiv (u_B(x) \pm u_A(x)) / \sqrt{2}$

$$\Rightarrow \psi(x,0) = u_B(x) \approx \frac{1}{\sqrt{2}} (u_S(x) + u_A(x))$$

$$\Rightarrow \psi(x,t) = \frac{1}{\sqrt{2}} (e^{-iE_S t/\hbar} u_S(x) + e^{-iE_A t/\hbar} u_A(x)) = \frac{e^{-iE_S t}}{\sqrt{2}} (u_S(x) + e^{-\frac{i\Delta E t}{\hbar}} u_A(x))$$

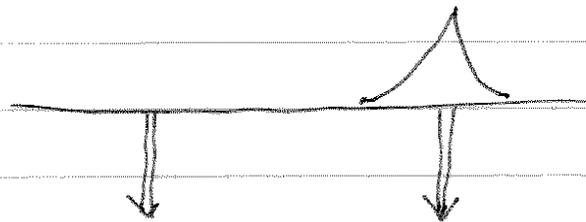
where $\Delta E = E_A - E_S$

$$\Rightarrow |\psi(x,t)|^2 = \frac{1}{2} |u_S(x) + e^{-\frac{i\Delta E t}{\hbar}} u_A(x)|^2$$

Thus $|\psi(x,t)|^2$ oscillates periodically. When

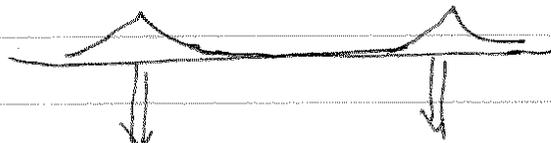
$$\frac{\Delta E t}{\hbar} = 2\pi, \quad |\psi(x,t)|^2 = |\psi(x,0)|^2 \Rightarrow \text{Period } T = 2\pi \frac{\hbar}{\Delta E}$$

$t=0$



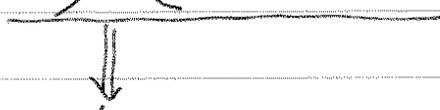
$$|\psi(x,0)|^2 = \frac{1}{2} |u_S(x) + u_A(x)|^2 = |u_B(x)|^2$$

$t=T/4$



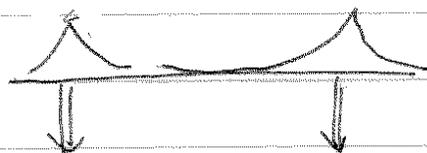
$$|\psi(x, T/4)|^2 = \frac{1}{2} |u_S(x) - i u_A(x)|^2 \approx \frac{1}{2} |u_B(x)|^2 + \frac{1}{2} |u_A(x)|^2$$

$t=T/2$



$$|\psi(x, T/2)|^2 = \frac{1}{2} |u_S(x) - u_A(x)|^2 = |u_A(x)|^2$$

$t=3T/4$



$$|\psi(x, 3T/4)|^2 = \frac{1}{2} |u_S(x) + i u_A(x)|^2 \approx \frac{1}{2} (|u_S|^2 + |u_A|^2)$$

$t=T$

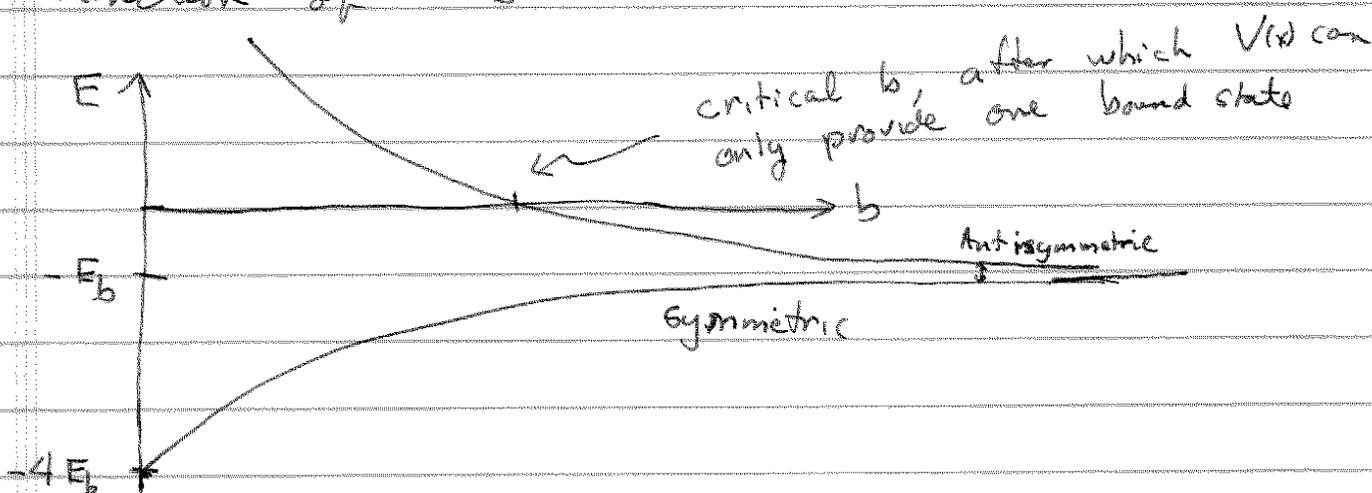


$$|\psi(x, T)|^2 = \frac{1}{2} |u_S(x) + u_A(x)|^2 = |u_B(x)|^2$$

(e) In the limit $b \rightarrow 0$ we have the effective potential $V(x) = -(2U_0) \delta(x)$. This potential possesses one bound state with binding energy $\frac{m(2U_0)^2}{2\hbar^2} = 4 \left(\frac{mU_0^2}{2\hbar^2} \right) = 4E_b$

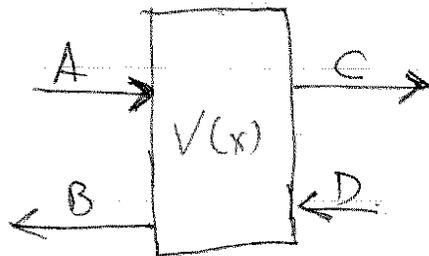
From part (c) in the limit $b \rightarrow \infty$ we have two degenerate bound states with binding energy E_b .

With these facts we sketch these eigenvalues as a function of b



Note: This is a ~~very~~ simple model of a H_2^+ molecule (two protons bound by one ^{electron} molecule). In the limit that the two nuclei are very far apart $R \gg a_0$ (Bohr radius) the electron eigenstates are degenerate and go to that of separated hydrogen atoms. In the limit $R \rightarrow 0$ we must go to the eigenstates of a He^+ ion.

Problem 3: The S-matrix



$$|\psi_{in}\rangle \doteq \begin{bmatrix} A \\ D \end{bmatrix}$$

$$|\psi_{out}\rangle \doteq \begin{bmatrix} B \\ C \end{bmatrix}$$

$$|\psi_{out}\rangle = \hat{S} |\psi_{in}\rangle$$

$$S \doteq \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \doteq \text{"S-matrix"}$$

(a) For a closed quantum system ($V(x)$ real)

S is a unitary matrix $\Rightarrow S^\dagger S = \mathbb{1}$

$$\Rightarrow \begin{bmatrix} |S_{11}|^2 + |S_{12}|^2 & S_{11} S_{21}^* + S_{12} S_{22}^* \\ S_{21} S_{11}^* + S_{22} S_{12}^* & |S_{22}|^2 + |S_{21}|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore |S_{11}|^2 + |S_{12}|^2 = 1 \quad |S_{22}|^2 + |S_{21}|^2 = 1$$

$$S_{11} S_{21}^* + S_{12} S_{22}^* = 0 \quad \checkmark$$

and trivially then the conjugate $S_{11}^* S_{21} + S_{12}^* S_{22} = 0$

(Subtract from $|S_{22}|^2 + |S_{21}|^2 = 1$) $|S_{11}|^2 - |S_{22}|^2 = |S_{21}|^2 - |S_{12}|^2 = 0$

$$\therefore |S_{11}| = |S_{22}| \quad |S_{21}| = |S_{12}| \quad \checkmark$$

Since \hat{S} is unitary its eigenvalues of unit mag complex numbers $\{e^{i\phi_1}, e^{i\phi_2}\}$, i.e. phases.

(b) Symmetric case

$$S = \begin{bmatrix} r & t \\ t & r \end{bmatrix}$$

r = reflection amplitude
 t = transmission amp.

Reflection coeff: $R = |r|^2$, $T = |t|^2$

From (a) $|S_{11}|^2 + |S_{12}|^2 = 1 \Rightarrow |r|^2 + |t|^2 = 1$
 $R + T = 1$

Thus $r = \sqrt{R} e^{i\phi_r}$ $t = |t| e^{i\phi_t} = \sqrt{1-R} e^{i\phi_t}$

Now $S_{11} S_{21}^* + S_{12} S_{22}^* = 0 \Rightarrow r t^* + t r^* = 0$

$$\Rightarrow 2|r||t| \cos(\phi_r - \phi_t) = 0$$

$$\Rightarrow \boxed{\phi_r - \phi_t = \pm \frac{\pi}{2}}$$

Transmission and reflection phase differ by $\pm \pi/2$

Now, $S = r \hat{1} + t \hat{\sigma}_x$

$$\Rightarrow \text{eigenvalues of } S = r \pm t \quad (\text{Since eigenvalues of } \hat{\sigma}_x = \pm 1)$$

$$\therefore e^{i\phi_1} = r + t \quad e^{i\phi_2} = r - t$$

$$\Rightarrow r = \frac{e^{i\phi_1} - e^{i\phi_2}}{2} \quad t = \frac{e^{i\phi_1} + e^{i\phi_2}}{2}$$

$$\boxed{r = i e^{i\bar{\phi}} \sin \Delta\phi \quad t = e^{i\bar{\phi}} \cos \Delta\phi}$$

$$\text{where } \bar{\phi} \equiv \frac{\phi_1 + \phi_2}{2}$$

$$\Delta\phi \equiv \phi_1 - \phi_2$$

(b continued)

$$R = |r|^2 = \sin^2(\phi_1 - \phi_2)$$

$$T = |t|^2 = \cos^2(\phi_1 - \phi_2)$$

$$\phi_r = \text{Arg}(r) = \frac{\phi_1 + \phi_2}{2} + \frac{\pi}{2}$$

This makes sense. When $\phi_1 = \phi_2$ $S = e^{i\phi_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

\Rightarrow No reflection, transmission \equiv phase shift $e^{i\phi_1}$

(c) We consider now scattering from an attractive delta function potential. We can find the r and t coeffs easily using the boundary conditions.

$$\begin{array}{c} e^{ikx} \text{ inc} \\ \leftarrow e^{-ikx} \text{ ref} \quad \rightarrow t e^{ikx} \text{ trans} \\ \hline \downarrow \\ x=0 \end{array}$$

$$\psi(0) \text{ continuous} \Rightarrow t = 1 + r$$

$$\psi'(0_-) - \psi'(0_+) = 2\gamma\psi(0) \Rightarrow ik(1 - r - t) = 2\gamma t$$

$$\text{Substitute for } r \Rightarrow 2ik(1 - t) = 2\gamma t$$

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$$\text{Thus } (\gamma + ik)t = ik$$

$$\Rightarrow \boxed{t = \frac{ik}{\gamma + ik}}$$

$$\boxed{r = t - 1 = \frac{-\gamma}{\gamma + ik}}$$

$$|r|^2 + |t|^2 = 1 \quad |\phi_t - \phi_r| = \frac{\pi}{2} \quad \checkmark$$

$$|H_e| = \frac{k}{\sqrt{\gamma^2 + k^2}} = \cos(\phi_1 - \phi_2)$$

$$\Rightarrow \phi_1 - \phi_2 = \cos^{-1}\left(\frac{k}{\sqrt{k^2 + \gamma^2}}\right)$$

$$\frac{\phi_1 + \phi_2}{2} = \text{Arg}\left(\frac{k}{\gamma + ik}\right) = -\text{Arg}(\gamma + ik) = \tan^{-1}\left(\frac{k}{\gamma}\right)$$

$$\Rightarrow \boxed{\begin{aligned} \phi_1 &= \frac{1}{2} \cos^{-1}\left(\frac{k}{\sqrt{k^2 + \gamma^2}}\right) + \tan^{-1}\left(\frac{k}{\gamma}\right) \\ \phi_2 &= -\frac{1}{2} \cos^{-1}\left(\frac{k}{\sqrt{k^2 + \gamma^2}}\right) + \tan^{-1}\left(\frac{k}{\gamma}\right) \end{aligned}}$$

Now $\gamma \rightarrow 0$ (No potential)

$$\phi_1 = \phi_2 = \frac{\pi}{2}$$

$$\phi_t = 0$$

(d) There are poles (singularities) in the S-matrix @

$$k = -i\gamma$$

$$\Rightarrow E = \frac{\hbar^2 k^2}{2m} \Rightarrow -\frac{\hbar^2 \gamma^2}{2m} = -\frac{m}{2\hbar^2} U_0$$

= Bound state energy!

The relationship between poles of the S-matrix and boundstates of the potential is an important general property that we will see again.