

Problem 1: Cold atoms in an "optical lattice"

Given: "Light-shift" interaction (AC Stark Shift)

Atomic resonance: two levels $|g\rangle$: ground state
 $|e\rangle$: excited state

$$\omega_{eg} = \frac{E_e - E_g}{\hbar} \quad \begin{array}{c} \uparrow \\ \Delta \\ \downarrow \end{array} \quad |g\rangle \quad |e\rangle \quad \Delta = \omega_e - \omega_g \quad (\text{detuning})$$

resonance frequency

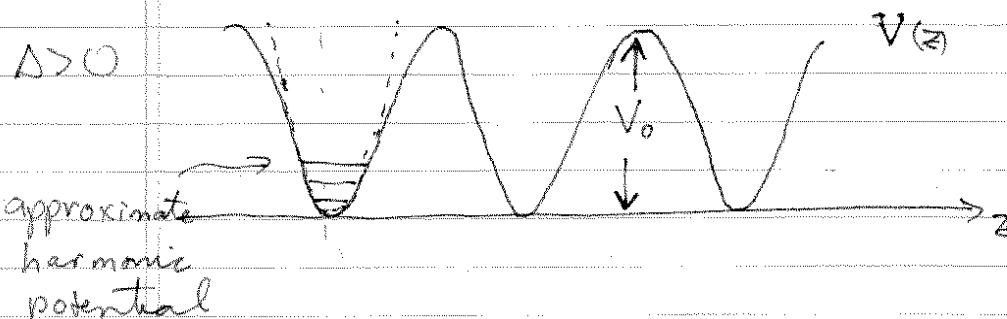
I_s = "saturation intensity"

$$\Gamma = \frac{1}{\text{Rabi time of } |e\rangle}$$

$$\hat{V}(x) = \frac{\hbar \Gamma^2}{8\Delta} \frac{I(x)}{I_s} \quad I(x) = \text{intensity distribution}$$

^{133}Cs atoms in a standing wave $I = I_0 \sin^2(k_z z)$

$$\Rightarrow \hat{V}(z) = \left(\frac{\hbar \Gamma^2}{8\Delta} \frac{I_0}{I_s} \right) \sin^2 k_z z \equiv V_0 \sin^2 k_z z$$



^{133}Cs parameters: $\frac{\Gamma}{2\pi} \approx 5 \text{ MHz}$

"D2 resonance"
 ${}^6S_{1/2} \leftrightarrow {}^6P_{3/2}$

$$I_s \approx 1 \text{ mW/cm}^2$$

$$\lambda_{eg} = 852 \text{ nm}$$

Q When is the atom trapped?

If we ignore tunneling between different "lattice sites" (i.e. neighboring local potentials) we must have

$$E \lesssim V_0 = \frac{\hbar \Gamma^2}{8\Delta} \frac{I_0}{I_s}$$

Given laser parameters with

$$\Delta = 5\Gamma, \quad I_0 = 1 \text{ mW/cm}^2 = I_s$$

$$V_0 = \frac{1}{40} \frac{\hbar \Gamma}{\Delta} = \frac{1}{40} \left(\frac{\hbar \Gamma}{2\pi} \right) = 8.3 \times 10^{-22} \text{ erg}$$

Approximate temperature: $k_B T \sim V_0$ (Boltzmann's constant)
 $k_B = 1.38 \times 10^{-16} \text{ erg/Kelvin}$

Require $\Rightarrow T \lesssim 6 \times 10^{-6} \text{ K} = 6 \mu\text{K}$

Such cold temperatures can be achieved with laser cooling. In fact, through a process known as "Sisyphus cooling" the optical lattice cools as well as traps.

For a review see:

P.S. Jessen and I.H. Deutsch, "Adv Atomic, Molecular and Optical Physics", Vol 37, 95 (1996)

Let us suppose $\delta k \Delta z_0 \equiv \eta \ll 1$

"Lamb-Dicke parameter"

where $\Delta z_0 = \frac{z_c}{\sqrt{2}} = \sqrt{\frac{\hbar}{2M\omega_{sc}}} = \text{Ground state rms}$

Let us rewrite the operator in terms of creation and annihilation: $\hat{z} = z_c (\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}})$

$$\begin{aligned}\exp\{i\delta k \hat{z}\} &= \exp\left\{i\delta k z_c \left(\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}\right)\right\} \\ &= \exp\left\{i\delta k \Delta z_0 (\hat{a} + \hat{a}^\dagger)\right\} = \exp\{i\eta(\hat{a} + \hat{a}^\dagger)\}\end{aligned}$$

For $\eta \ll 1$

$$\exp\{i\delta k \hat{z}\} = \exp\{i\eta(\hat{a} + \hat{a}^\dagger)\} \approx 1 + i\eta(\hat{a} + \hat{a}^\dagger)$$

$$\Rightarrow P_{0 \rightarrow n} \propto |\langle n | e^{i\delta k \hat{z}} | 0 \rangle|^2 \propto |\langle n | 0 \rangle + i\eta \langle n | \hat{a} + \hat{a}^\dagger | 0 \rangle|^2$$

$$\text{Ansatz: } \langle n | 0 \rangle = \delta_{n,0}$$

$$\langle n | \hat{a} + \hat{a}^\dagger | 0 \rangle = \langle n | 1 \rangle = \delta_{n,1}$$

$$\Rightarrow P_{0 \rightarrow n} \propto |\delta_{n,0} + i\eta \delta_{n,1}|^2 = \delta_{n,0}^2 + \eta^2 \delta_{n,1}^2$$

Lamb-Dicke effect: $\delta k \Delta z_0 \ll 1 \Rightarrow$ Probability of scattering photon and changing vibrational level is suppressed by factor η^2

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(c) The quantum nature of the motion becomes important when the temperature is comparable to the energy level spacing:

$$\Rightarrow k_B T_Q \sim \hbar \omega_{osc} = 2 \sqrt{V_0 E_R}$$

$$\Rightarrow T_Q \sim 2 \sqrt{\frac{V_0}{k_B} \left(\frac{E_R}{\hbar} \right)} = 2 \sqrt{(6 \text{ eV})(0.1 \mu\text{K})}$$

$$\Rightarrow \boxed{T_{\text{Quantum}} \sim 1.5 \mu\text{K}}$$

The mechanical frequency:

$$\nu_{osc} = \frac{\omega_{osc}}{2\pi} = 2 \sqrt{\frac{V_0}{h} \frac{E_R}{\hbar}} \quad \cancel{\text{Hz}}$$

Aside: $\frac{V_0}{h} = \frac{1}{40} \left(\frac{1}{2\pi} \right) = 125 \text{ kHz}$

$$\frac{E_R}{\hbar} = 2.0 \text{ kHz}$$

$$\Rightarrow \boxed{\nu_{osc} = 31 \text{ kHz}} \text{ radio frequency}$$

(d) If an atom scatters a photon the recoil momentum can cause the atom to change its vibrational energy level. The probability of a transition $|n\rangle \rightarrow |n'\rangle$ is proportional to

$$P_{n \rightarrow n'} \sim |\langle n' | e^{i \vec{s} \cdot \hat{k}} | n \rangle|^2$$

where $\vec{s} \cdot \hat{k} = \vec{e}_z \cdot (\vec{k}_f - \vec{k}_i)$ (Change in wave vector along z)

(c) If $\Delta k = k_L$ then

$$\gamma = k_L \Delta z_0 = k_L \sqrt{\frac{\hbar}{2m\omega_{osc}}} = \sqrt{\frac{\hbar^2 k_L^2}{2M\omega_{osc}}}$$

$$\Rightarrow \gamma = \sqrt{\frac{E_R}{\hbar\omega_{osc}}} = \frac{1}{\sqrt{2}} \left(\frac{E_R}{V_0} \right)^{1/4} = 0.25$$

$$\Rightarrow \Delta z_0 = \frac{1}{4} \chi_L = \frac{\lambda_L}{25}$$

Atom in ground
vibrational level located
much better than λ

Relative probability:

$$\boxed{\frac{P_{0 \rightarrow 1}}{P_{0 \rightarrow 0}} = \gamma^2 = 0.06}$$

Lamb-Dicke effect:

A physical picture of the Lamb-Dicke effect is as follows. Absorption and emission of a photon gives the atom a photon recoil $p_{\text{recoil}} = \hbar \delta k$

The effect of this recoil on the state is momentum translation:

$$|n\rangle \Rightarrow e^{ip_{\text{recoil}}^2/2\hbar} |n\rangle = e^{i\Delta k^2/2} |n\rangle$$

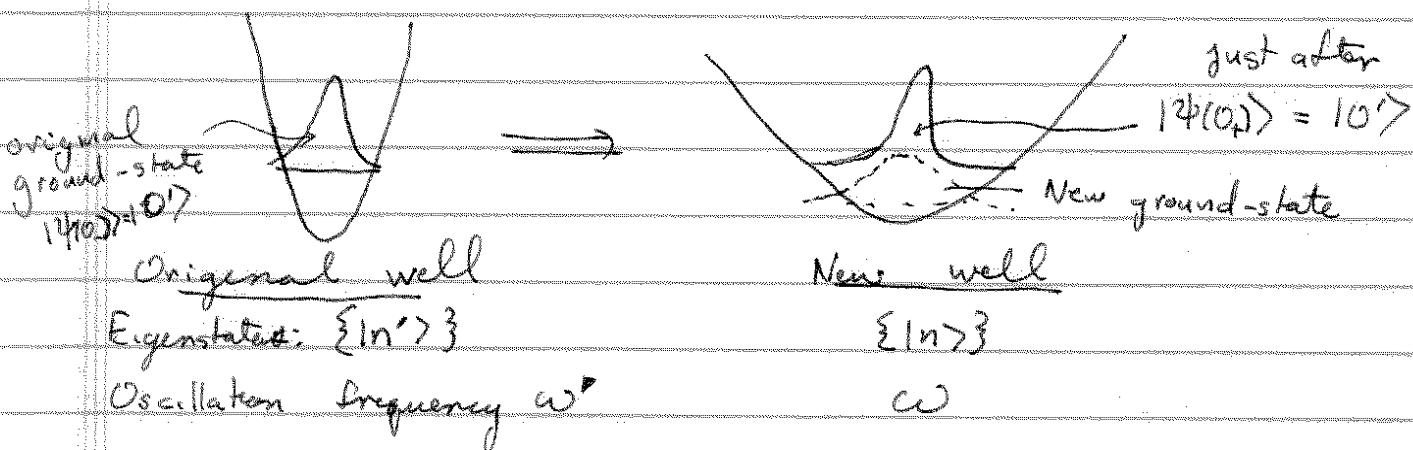
However if Δk is small (well localized) there is a large momentum uncertainty and the effect of the recoil is negligible

$$\Rightarrow e^{i\Delta k^2/2} |n\rangle \approx |n\rangle$$

Thus the translated state is still approximately orthogonal to the other energy eigenstates and no transition occurs.

Problem 3+4 - Breathing wavepackets

Sudden change in well curvature



Given: At $t=0$ $|\psi(0)\rangle = |0\rangle$ (original ground state)

At $t=t_0^+$, sudden change \Rightarrow wave function the same

$$|\psi(t_0^+)\rangle = |\psi(0)\rangle = |0\rangle$$

This state is no longer a stationary state and will evolve non-trivially with time.

- Condition for "sudden approximation"

If Γ = rate of change of potential

Require $\Gamma \gg$ Bohr frequency = ω' (in this case)

- Evolution for $t > t_0^+$

$$|\psi(t_0^+)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \text{ where } c_n = \langle n | \psi(0) \rangle$$

$$\Rightarrow |\psi(t)\rangle = \sum c_n e^{-iE_n t/\hbar} |n\rangle = e^{-i\omega t_0^+} \sum c_n e^{-i\omega t} |n\rangle$$

Now, in the initial decomposition

$c_n = \langle n | \psi(0) \rangle = 0$ if n odd
since $|\psi(0)\rangle$ is a state of even parity
and the energy eigenstates has parity $(-1)^n$

$$\Rightarrow |\psi(+)\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} c_{2n} e^{-i2\omega t} |n\rangle$$

neglect

Period of oscillation: T

Smallest T such that: $|\psi(t+T)\rangle = \psi(+)|\psi(+)\rangle$ (up to a phase)

$$\Rightarrow \sum_n c_{2n} e^{-i2\omega t} e^{-i2\omega T} |n\rangle = \sum_n c_{2n} e^{-i2\omega t} |n\rangle$$

$$\Rightarrow T = \frac{2\pi}{(2\omega)} = \frac{\pi}{\omega} \Rightarrow \boxed{\text{Oscillation frequency } 2\omega}$$

Note: This conclusion depends crucially on the equal spacing of the energy levels

If the potential were anharmonic then the different eigenstates would not all be "harmonics" of the fundamental and the oscillation would dephase. However because of the discrete set of modes (eigenstates) there will always be a recurrence time to get back to the initial wave form (see problem set 5, 1d), given by $1/\Delta E_{\text{Bohr-m}}^2$

where $\Delta E_{\text{Bohr-m}}$ is the minimum energy spacing.

(b) Now for a quantitative description:

Given: $|0'\rangle = \hat{S}(r)|0\rangle$, $\hat{S}(r) = e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{+2})}$
 (unitary operator)

Consider:

$$\hat{S}(r)^+ \hat{a} \hat{S}(r) = \sum_n [\hat{A}, \hat{a}]^{(n)} \frac{1}{n!} \quad (\text{Baker-Hausdorff})$$

$$\text{where } \hat{A} = \frac{r}{2}(\hat{a}^{+2} - \hat{a}^2)$$

$$[\hat{A}, \hat{a}]^{(0)} = \hat{a}$$

$$[\hat{A}, \hat{a}]^{(1)} = \frac{r}{2} [\hat{a}^{+2}, \hat{a}] = \frac{r}{2} (\hat{a}^{+} [\hat{a}^{+}, \hat{a}] + [\hat{a}^{+}, \hat{a}] \hat{a}^{+}) \\ = -r \hat{a}^{+}$$

$$[\hat{A}, \hat{a}]^{(2)} = [\hat{A}, -r \hat{a}^{+}] = \frac{r^2}{2} [\hat{a}^2, \hat{a}^{+}] = \frac{r^2}{2} (-[\hat{a}^{+2}, \hat{a}]) \\ = r^2 \hat{a}$$

etc. Thus we have the pattern

$$[\hat{A}, \hat{a}]^{(n)} = \begin{cases} r^n \hat{a} & n \text{ even } 0, 2, 4, \dots \\ -r^n \hat{a}^{+} & n \text{ odd } 1, 3, 5, \dots \end{cases}$$

$$\Rightarrow \hat{S}(r)^+ \hat{a} \hat{S}(r) = \left(\sum_{\text{even}} \frac{r^n}{n!} \right) \hat{a} - \left(\sum_{\text{odd}} \frac{r^n}{n!} \right) \hat{a}^{+}$$

$$\boxed{\hat{S}(r)^+ \hat{a} \hat{S}(r) = (\cosh r) \hat{a} - (\sinh r) \hat{a}^{+}}$$

This is known as a Bogoliubov transformation

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For short hand let $\begin{cases} c = \cosh(r) \\ s = \sinh(r) \end{cases} \Rightarrow \begin{cases} \hat{S}^+ \hat{a} \hat{S} = c \hat{a} - s \hat{a}^{+} \\ \hat{S}^+ \hat{a}^{+} \hat{S} = c \hat{a}^{+} - s \hat{a} \end{cases}$

$$\text{with } \hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \quad \hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}$$

$$\hat{S}_m^+ \hat{X} \hat{S}(r) = \frac{1}{\sqrt{2}} (\hat{S}^+ \hat{a} \hat{S}) + \frac{1}{\sqrt{2}} (\hat{S}^+ \hat{a}^\dagger \hat{S})$$

$$= \frac{1}{\sqrt{2}} (c\hat{a} - s\hat{a}^\dagger) + \frac{1}{\sqrt{2}} (c\hat{a}^\dagger - s\hat{a})$$

$$= \frac{c-s}{\sqrt{2}} \left(\hat{a} + \hat{a}^\dagger \right)$$

Recall
 $c = e^{-r} + e^{+r}$

$$s = \frac{e^{-r} - e^{+r}}{2}$$

$$\Rightarrow \boxed{\hat{S}_m^+ \hat{X} \hat{S}(r) = e^{-r} \hat{X}}$$

Similarly:

$$\hat{S}_m^+ \hat{P} \hat{S}(r) = \frac{1}{i\sqrt{2}} (\hat{S}^+ \hat{a} \hat{S}) - \frac{1}{i\sqrt{2}} (\hat{S}^+ \hat{a}^\dagger \hat{S})$$

$$= \frac{1}{i\sqrt{2}} (c\hat{a} - s\hat{a}^\dagger) - \frac{1}{i\sqrt{2}} (c\hat{a}^\dagger - s\hat{a})$$

$$= (c+s) \left(\hat{a} - \frac{\hat{a}^\dagger}{i\sqrt{2}} \right)$$

$$\Rightarrow \boxed{\hat{S}_m^+ \hat{P} \hat{S}(r) = e^{+r} \hat{P}}$$

$$\langle 0' | \hat{X} | 10' \rangle = \langle 01 | \hat{S}_m^+ \hat{X} \hat{S}(r) | 10 \rangle = e^{-r} \langle 01 | \hat{X} | 10 \rangle = 0$$

$$\langle 0' | \hat{P} | 10' \rangle = \langle 01 | \hat{S}_m^+ \hat{P} \hat{S}(r) | 10 \rangle = e^{+r} \langle 01 | \hat{P} | 10 \rangle = 0$$

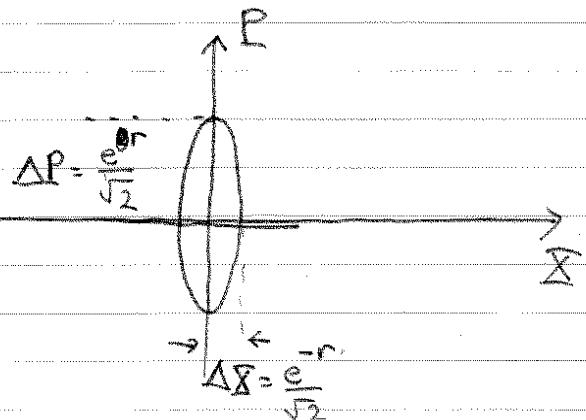
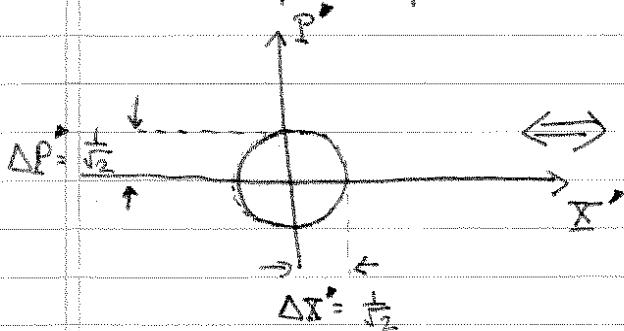
$$\Rightarrow \Delta \hat{X}^2 = \langle 0' | \hat{X}^2 | 10' \rangle = \langle 01 | \hat{S}_m^+ \hat{X}^2 \hat{S}(r) | 10 \rangle = \langle 01 | (\hat{S}^+ \hat{X} \hat{S})^2 | 10 \rangle$$

$$\Rightarrow \boxed{\Delta \hat{X}^2 = e^{-2r} \langle 01 | \hat{X}^2 | 10 \rangle = \frac{1}{2} e^{-2r}}$$

$$\Delta \hat{P}^2 = \langle 0' | \hat{P}^2 | 10' \rangle = \langle 01 | \hat{S}^+ \hat{P}^2 \hat{S} | 10 \rangle = \langle 01 | (\hat{S}^+ \hat{P} \hat{S})^2 | 10 \rangle$$

$$\Rightarrow \boxed{\Delta \hat{P}^2 = e^{+2r} \langle 01 | \hat{P}^2 | 10 \rangle = \frac{1}{2} e^{+2r}}$$

(b) Phase-space picture



Stationary ground-state

"Vacuum"

"Squeezed ground-state"

"Squeezed vacuum"

Let
(c) Consider $|\psi_1\rangle \equiv \hat{S}(r)|\Xi\rangle$

$$|\psi_2\rangle \equiv \hat{S}(r)|\Xi_2\rangle$$

where

$$\hat{\Xi}|\Xi_1\rangle = \Xi_1|\Xi_1\rangle$$

$$\hat{\Xi}|\Xi_2\rangle = \Xi_2|\Xi_2\rangle$$

Consider:

$$\langle \psi_2 | \hat{\Xi} | \psi_1 \rangle = \langle \Xi_2 | \hat{S}^\dagger(r) \hat{\Xi} \hat{S}(r) | \Xi_1 \rangle = e^{-r} \langle \Xi_2 | \hat{\Xi} | \Xi_1 \rangle$$

$$= e^{-r} \Xi_1 \delta(\Xi_1 - \Xi_2)$$

$$= \Xi'_1 \underbrace{\delta(\Xi'_1 - \Xi'_2)}_{e^{-r}}$$

where

$$\Xi'_1 = e^{-r} \Xi_1$$

$$\therefore |\psi\rangle = \hat{S}(r)|\Xi\rangle = \frac{1}{\sqrt{e^{-r}}} |e^{-r}\Xi\rangle$$

and $\delta(ax) = \frac{1}{a} \delta(x)$

$$\begin{aligned} \text{Note also } \hat{S}^\dagger(r)|\Xi\rangle &= \hat{S}(-r)|\Xi\rangle = \hat{S}(-r)|\Xi\rangle \\ &= \frac{1}{\sqrt{e^r}} |e^r\Xi\rangle \end{aligned}$$

(The normalization factor was missing from the problem set statement)

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$$\text{Given: } \Psi(x, 0) = \langle x | \Psi(0) \rangle = \langle x | 0' \rangle = \frac{1}{(2\pi x_c^2)^{1/4}} e^{-\frac{x^2}{4x_c^2}}$$

$$= \frac{1}{\sqrt{\pi} x_c} e^{-\frac{1}{2} \left(\frac{x}{x_c}\right)^2}$$

where $x_c' = \sqrt{\frac{\hbar}{m\omega}}$ (w.r.t. original potential)

Now go to dimensionless variables with respect to new potential:

$$x_c' \equiv \sqrt{\frac{\hbar}{m\omega}}, \quad x_0 = \frac{x_c}{\sqrt{2}}, \quad X = \frac{x}{x_c}$$

$$\Psi(X, 0) = \langle X | \Psi(0) \rangle = \sqrt{x_c'} \Psi(x = x_c X, 0)$$

$$= \sqrt{\frac{x_c}{x_c'}} \frac{1}{\pi^{1/4}} e^{-\frac{X^2}{2} \left(\frac{x_c}{x_c'}\right)^2}$$

$$\text{Let } e^r = \frac{x_c}{x_c'} = \sqrt{\frac{\omega}{\omega}} \Rightarrow \langle X | \Psi(0) \rangle = \frac{e^{r/2}}{\pi^{1/4}} e^{-\frac{X^2 e^{2r}}{2}}$$

We want to show that

$$\langle \Psi(0) \rangle = |0\rangle = \hat{S}(r)|0\rangle$$

Check

$$\langle X | \Psi(0) \rangle = \langle X | \hat{S}(r)|0\rangle \quad \left(\begin{array}{l} \text{Ascle} \\ \langle X | \hat{S}(r) \Rightarrow \hat{S}^\dagger(r) |X\rangle \\ \text{---} \\ \hat{S}^\dagger(r) |X\rangle \Rightarrow \hat{S}(r) |X\rangle \end{array} \right)$$

$$= e^{r/2} \langle X e^r | 0 \rangle$$

$$= e^{r/2} U_0(X e^r)$$

$$= e^{r/2} \frac{1}{\pi^{1/4}} e^{-\frac{X^2 e^{2r}}{2}}$$

(d) The state for times $t > 0$ is given by

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle = \hat{U}(t)\hat{S}(r)|0\rangle$$

As if Problem 2 we insert $\hat{U}^\dagger(t)\hat{U}(t) = \hat{1}$

$$\Rightarrow |\Psi(t)\rangle = \hat{U}(t)\hat{S}(r)\hat{U}^\dagger(t)\underbrace{|\hat{U}(t)|0\rangle}_{= e^{-i\omega t/2}|0\rangle} \xrightarrow{\text{neglect zero-point phase}}$$

Axes:

$$\text{Consider: } \hat{U}(t)\hat{S}(r)\hat{U}^\dagger(t) = \hat{U}(t)\exp\left\{\frac{r}{2}(\hat{a}^2 - \hat{a}^{+2})\right\}\hat{U}^\dagger(t)$$

$$= \exp\left\{\frac{r}{2}(\hat{U}_0\hat{a}^2\hat{U}^\dagger_0 - \hat{U}_0\hat{a}^{+2}\hat{U}^\dagger_0)\right\}$$

$$= \exp\left\{\frac{r}{2}((\hat{U}\hat{a}\hat{U}^\dagger)^2 - (\hat{U}\hat{a}^+\hat{U}^\dagger)^2)\right\}$$

$$= \exp\left\{\frac{r}{2}((\hat{a}e^{i\omega t})^2 - (\hat{a}^+e^{-i\omega t})^2)\right\}$$

$$= \exp\left\{\frac{1}{2}(\zeta^*(t)\hat{a}^2 - \zeta(t)\hat{a}^{+2})\right\}$$

where $\zeta(t) = r e^{2i\omega t}$ (known as the "squeezing parameter")

$$\Rightarrow |\Psi(t)\rangle = \hat{S}(\zeta(t))|0\rangle$$

(time dependent "squeezed state")

(c) We now want to calculate $\langle \Delta \hat{x}^2 \rangle_t$ and $\langle \Delta \hat{p}^2 \rangle_t$.

Heisenberg picture:

$$|\psi\rangle = |0\rangle = \hat{S}(r)|0\rangle \quad (\text{constant in time})$$

$$\hat{x}(t) = \hat{U}^\dagger(t) \hat{x}(0) \hat{U}(t) = \hat{x}(0) \cos \omega t + \hat{p}(0) \sin \omega t$$

$$\hat{p}(t) = \hat{U}^\dagger(t) \hat{p}(0) \hat{U}(t) = +\hat{p}(0) \cos \omega t - \hat{x}(0) \sin \omega t$$

$$\Rightarrow \langle \hat{x} \rangle_t = \langle \psi | \hat{x}(0) | \psi \rangle = \langle \psi | \hat{x}(0) | \psi \rangle \cos \omega t + \langle \psi | \hat{p}(0) | \psi \rangle \sin \omega t = 0$$

$$\langle \hat{p} \rangle_t = \langle \psi | \hat{p}(0) | \psi \rangle \cos \omega t - \langle \psi | \hat{x}(0) | \psi \rangle \sin \omega t = 0$$

$$\begin{aligned} \therefore \langle \hat{x}^2 \rangle_t &= \langle \hat{x} \rangle_t^2 = \langle \psi | \hat{x}(t)^2 | \psi \rangle \\ &= \langle \psi | \hat{x}^2(0) | \psi \rangle \cos^2 \omega t + \langle \psi | \hat{p}(0)^2 | \psi \rangle \sin^2 \omega t \\ &\quad + (\langle \psi | \hat{x}(0) \hat{p}(0) | \psi \rangle + \langle \psi | \hat{p}(0) \hat{x}(0) | \psi \rangle) \frac{\cos \omega t \sin \omega t}{2} \end{aligned}$$

(no initial correlation in \hat{x} and \hat{p})

Using part (b)

$$\Rightarrow \boxed{\langle \Delta \hat{x}^2 \rangle_t = \frac{e^{-2r}}{2} \cos^2 \omega t + \frac{e^{2r}}{2} \sin^2 \omega t}$$

Similarly

$$\boxed{\langle \Delta \hat{p}^2 \rangle_t = \frac{e^{2r}}{2} \cos^2 \omega t + \frac{e^{-2r}}{2} \sin^2 \omega t}$$

On the next page I demonstrate the same calculation using the Schrödinger picture.

Schrödinger picture: Given arbitrary operator \hat{A}

$$\langle \hat{A} \rangle_t = \langle \psi(t) | \hat{A} | \psi(t) \rangle = \langle 0 | \hat{S}^\dagger(\xi(t)) \hat{A} \hat{S}(\xi(t)) | 0 \rangle$$

We must therefore consider the unitary transformation

$$\hat{S}^\dagger(\xi(t)) \hat{a} \hat{S}(\xi(t)) = \sum_n \frac{1}{n!} \left[\frac{\xi(t) \hat{a}^2 - \xi^* \hat{a}^2}{2}, \hat{a} \right]^{(n)}$$

Aside: $\left[\frac{\xi(t) \hat{a}^2 - \xi^* \hat{a}^2}{2}, \hat{a} \right] = \begin{cases} i\xi t^n \hat{a} & n \text{ even} \\ i\xi^{n-1} \xi^* \hat{a}^+ & n \text{ odd} \end{cases} = r^n \hat{a} - e^{2i\omega t} \hat{a}^+$

$$\Rightarrow \hat{S}^\dagger(\xi(t)) \hat{a} \hat{S}(\xi(t)) = \cosh r \hat{a} - e^{2i\omega t} \sinh r \hat{a}^+ = e^r \hat{a} - e^{-2i\omega t} \hat{a}^+$$

As before, we can easily show $\langle \hat{x} \rangle_t = \langle \hat{p} \rangle_t = 0$

$$\Rightarrow \langle \hat{x}^2 \rangle_t = \langle \hat{x}^2 \rangle_t = \frac{1}{2} \langle \psi(t) | \hat{a}^2 + \hat{a}^{+2} + \hat{a} \hat{a}^+ + \hat{a}^+ \hat{a} | \psi(t) \rangle$$

$$= \frac{1}{2} (\langle 0 | \hat{a}^2 | 0 \rangle + \langle 0 | \hat{a}^{+2} | 0 \rangle + \langle 0 | \hat{a} \hat{a}^+ | 0 \rangle + \langle 0 | \hat{a}^+ \hat{a} | 0 \rangle)$$

Aside:

- $\langle 0 | \hat{a}^2 | 0 \rangle = \langle 0 | (e^r \hat{a} - e^{-2i\omega t} \hat{a}^+)^2 | 0 \rangle$
- $= -e^{2ir} e^{2i\omega t} \langle 0 | \hat{a} \hat{a}^+ | 0 \rangle = -e^{2ir} e^{2i\omega t}$
- $= \langle 0 | \hat{a}^{+2} | 0 \rangle *$

- $\langle 0 | \hat{a}^+ \hat{a} | 0 \rangle = \langle 0 | (e^r \hat{a}^+ - e^{-2i\omega t} \hat{a}) (e^r \hat{a} - e^{2i\omega t} \hat{a}^+) | 0 \rangle$
- $= s^2$

- $\langle 0 | \hat{a} \hat{a}^+ | 0 \rangle = \langle 0 | (e^r \hat{a} - e^{2i\omega t} \hat{a}^+) (e^r \hat{a}^+ - e^{-2i\omega t} \hat{a}) | 0 \rangle$
- $= e^2$

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Putting this all together:

$$\langle \Delta X^2 \rangle_t = \frac{1}{2} (c^2 + \alpha^2 - 2c\alpha \cos 2\omega t)$$

Aside: $\begin{cases} c^2 + \alpha^2 = \frac{1}{2} (e^{2r} + e^{-2r}) \\ c\alpha = \frac{1}{4} (e^{2r} - e^{-2r}) \end{cases}$

$$\Rightarrow \langle \Delta X^2 \rangle_t = \frac{1}{4} \left\{ e^{2r} + e^{-2r} - (e^{2r} - e^{-2r}) \cos 2\omega t \right\}$$

$$\rightarrow \langle \Delta X^2 \rangle_t = \frac{1}{2} \left\{ e^{-2r} \left(\frac{1 + \cos 2\omega t}{2} \right) + e^{2r} \left(\frac{1 - \cos 2\omega t}{2} \right) \right\}$$

$$\rightarrow \boxed{\langle \Delta X^2 \rangle_t = \frac{1}{2} \left\{ e^{-2r} \cos^2 \omega t + e^{2r} \sin^2 \omega t \right\}}$$

^{A5}
before

Similarly:

$$\langle \Delta P^2 \rangle_t = \langle \psi(t) | \left(\frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2i}} \right)^2 | \psi(t) \rangle = \langle 0 | \left(\frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2i}} \right)^2 | 0 \rangle$$

$$= \frac{1}{2} \langle 0 | \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger - \hat{a}^2 - \hat{a}^{+2} | 0 \rangle$$

$$= \frac{1}{2} (2c\alpha \cos 2\omega t + c^2 + \alpha^2)$$

$$= \frac{1}{4} ((e^{2r} - e^{-2r}) \cos 2\omega t + e^{2r} + e^{-2r})$$

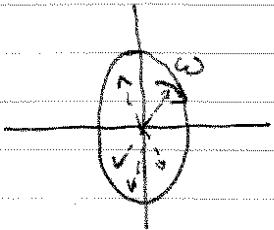
$$= \frac{1}{2} \left(e^{2r} \left(\frac{1 + \cos 2\omega t}{2} \right) + e^{-2r} \left(\frac{1 - \cos 2\omega t}{2} \right) \right)$$

$$\rightarrow \boxed{\langle \Delta P^2 \rangle_t = \frac{1}{2} \left\{ e^{2r} \cos^2 \omega t + e^{-2r} \sin^2 \omega t \right\}}$$

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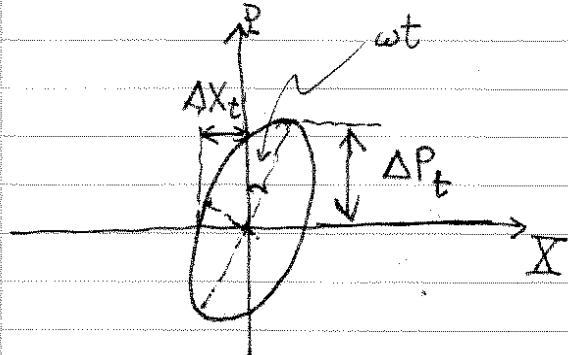
The evolution of the variances in \hat{X} and \hat{P} can be easily understood using the phase-space picture

At $t=0$ we have "squeezed vacuum"



We can think of this "error-ellipse" as representing a "statistical distribution of phasors, rotating clockwise with frequency ω

At a later time $t > 0$



Simple geometry shows

$$\Delta X_t^2 = \Delta X_0^2 \cos^2 \omega t + \Delta P_0^2 \sin^2 \omega t$$

$$\Delta P_t^2 = \Delta P_0^2 \cos^2 \omega t + \Delta X_0^2 \sin^2 \omega t$$

$$\text{where } \Delta X_0^2 = \frac{P}{2}^{-2n}, \quad \Delta P_0^2 = \frac{e}{2}^{+2n}$$

From this picture it is easy to see that the error-ellipse returns to its original orientation in $\frac{1}{2}$ the period of an phasor: i.e. packet breathes with frequency 2ω .

Squeezed states are important in quantum optics, providing new possibilities for precision measurement by reducing quantum uncertainties for certain observables (of course quantum uncertainty is increased along another observable, consistent with the ~~uncertainty~~ principle)

For a nice review see: R. Loudon + P.L. Knight, "Squeezed Light", J. Mod. Opt. 34, 709 (1987)

For application to precision interferometry see:
C.M. Caves, Phy Rev D 23 (1981)

Extra Credit:

Decomposition in number states

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$c_n = \langle n | \psi(0) \rangle$$

Using Hermite Polynomials

In the dimensionless units used in class

$$\Psi(\bar{x}) = \sqrt{x_c} \psi(x=x_c \bar{x})$$

$$\text{where } x_c = \sqrt{\frac{\hbar}{m\omega}}$$

$$\Rightarrow U_n(\bar{x}) = A_n H_n(\bar{x}) e^{-\bar{x}^2/2} \quad (\text{energy eigenstates})$$

$$\Rightarrow c_n = \int_{-\infty}^{\infty} d\bar{x} U_n(\bar{x}) \Psi(\bar{x}, 0)$$

for the case at hand

$$\psi(x, 0) = \frac{1}{(2\pi x_0)^{1/2}} e^{-\frac{x^2}{4x_0^2}}, \quad \text{where } x_0 = \sqrt{\frac{\hbar}{2m\omega}} \\ = \frac{x_c}{\sqrt{2}}$$

\Rightarrow In the characteristic units of the new potential

$$\Psi(\bar{x}, 0) = \sqrt{\frac{x_c}{x_0}} \frac{1}{\pi^{1/4}} e^{-\frac{\bar{x}^2}{2} \left(\frac{x_c}{x_0}\right)^2}$$

Using the definition of the squeezing parameter

$$e^{+r} = \frac{x_c}{x_0} \Rightarrow \Psi(\bar{x}, 0) = \frac{e^{r/2}}{\pi^{1/4}} e^{-\frac{\bar{x}^2 e^{2r}}{2}}$$

Thus

$$c_n = \int_{-\infty}^{\infty} dx H_n(x) \Psi(x, 0)$$

$$= \frac{A_n e^{r/2}}{\pi^{1/4}} \int_{-\infty}^{\infty} dx H_n(x) \exp\left\{-\frac{x^2}{2}(1+e^{+2r})\right\}$$

We must therefore evaluate an integral of the form

$$I_n = \int_{-\infty}^{\infty} dx H_n(x) e^{-ax^2} \quad a = \frac{1+e^{+2r}}{2} = e^r \cosh r$$

Here's one approach:

Use the "Generating Function": GT, B_V, Eq. (II) ^{complement}

$$e^{-x^2 + 2\lambda x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} H_n(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} dx e^{-x^2 + 2\lambda x - ax^2} = \sum_{n=0}^{\infty} \frac{2^n}{n!} \int_{-\infty}^{\infty} dx H_n(x) e^{-ax^2}$$

Now, the left hand side can be done (completing the square, etc., as we did in problem set #4)

$$\sqrt{a} e^{(\frac{1-a}{a})\lambda^2} = \sum_{n=0}^{\infty} \frac{2^n}{n!} I_n$$

Expand the left hand side in a power series for $\lambda \ll 1$

$$\Rightarrow \sqrt{a} \sum_{m=0}^{\infty} \frac{(1-a)^m \lambda^{2m}}{m!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} I_n$$

We can now equate the two series term by term to find

$$I_n = 0 \quad n \text{ odd} \quad (\text{Good, we expected this from parity considerations})$$

Relabeling $n=2m$ on left hand side

$$\sqrt{\pi} \sum_{m=0}^{\infty} \left(\frac{(-a)}{a}\right)^m \frac{\gamma^{2m}}{m!} = \sum_{m=0}^{\infty} \frac{\gamma^{2m}}{(2m)!} I_{2m}$$

$$\Rightarrow I_{2m} = \sqrt{\pi} \left(\frac{(-a)}{a}\right)^m (-1)^m H_{2m}(0)$$

$$\text{Using } H_{2m}(0) = (-1)^{\frac{(2m)!}{m!}} \quad (\text{easy to show})$$

$$\Rightarrow I_n = \begin{cases} 0 & n \text{ odd} \\ \sqrt{\pi} \left(\frac{(-a)}{a}\right)^{n/2} H_n(0) & n \text{ even} \end{cases}$$

Note $H_n(0) = 0$ for n odd so the condition is unnecessary.

Now let's put it all together

We have

$$C_n = \frac{A_n e^{r/2}}{\pi^{1/4}} \quad I_n = \frac{\pi^{1/4} A_n}{\sqrt{e^r}} \left(\frac{a-1}{a}\right)^{n/2} H_n(0)$$

$$\text{where } A_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \quad a = \frac{1+e^{+2r}}{2}$$

$$\text{Ansatz: } \frac{a-1}{a} = \frac{-1+e^{2r}}{2} = \frac{e^r - e^{-r}}{e^r + e^{-r}} = \frac{\sinh r}{\cosh r} = \frac{v}{u}$$

$$e^{-r} u = \cosh r = u$$

where $\mu = \cosh r$ $v = \sinh r$

$$\Rightarrow C_n = \frac{\pi^{1/4} A_n}{\sqrt{\mu}} \left(\frac{2u}{\mu}\right)^{n/2} H_n(0)$$

$$\Rightarrow [C_n = \frac{1}{\sqrt{n! \mu}} \left(\frac{v}{2\mu}\right)^{n/2} H_n(0)] \quad \text{where!}$$

$$= \left\{ \frac{1}{\sqrt{n! \mu}} \left(\frac{v}{2\mu}\right)^{n/2} \frac{n!}{(n/2)!} (-1)^{n/2} \quad n \text{ even} \right.$$

{ 0 odd

Now we seek $\Psi(x, t)$. According to the standard problem procedure, if

$$\Psi(x, 0) = \sum_{n=0}^{\infty} c_n U_n(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n! \mu}} \left(\frac{v}{2\mu}\right)^{n/2} A_n(0) U_n(x)$$

then

$$\begin{aligned} \Psi(x, 0) &= \sum_{n=0}^{\infty} c_n e^{-i\omega nt} U_n(x) e^{-i\omega t/2} \xrightarrow{\text{Zero point phase}} \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n! \mu}} \left(\frac{v e^{-2i\omega t}}{2\mu}\right)^{n/2} H_n(0) U_n(x) \end{aligned}$$

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Thus we arrive at the time development by mapping

$$\text{and } v \rightarrow v e^{-2i\omega t} \quad \text{or} \quad \sinh(r) \rightarrow \sinh(r) e^{-2i\omega t}$$

$$\mu \rightarrow \mu \quad \cosh(r) \rightarrow \cosh(r)$$

(this is the evolution we found using the operators)

Let us write $\Phi(x,0)$ in terms of μ and v

$$e^r = \mu + v \quad \bar{e}^r = \mu - v$$

$$\Rightarrow \Phi(x,0) = \frac{1}{\pi^{1/4} (\mu - v)^{1/2}} \exp \left\{ -\frac{x^2}{2} \frac{1}{(\mu - v)} \right\}$$

$$\therefore \Phi(x,t) = \frac{1}{\pi^{1/4} (\mu - v e^{-2i\omega t})^{1/2}} \exp \left\{ -\frac{x^2}{2(\mu - v e^{-2i\omega t})} \right\}$$

$$\begin{aligned} \text{As we: } \mu + v e^{-2i\omega t} &= \cosh(r) + e^{-2i\omega t} \sinh(r) \\ &= \frac{\bar{e}^r}{2} (1 + e^{-2i\omega t}) + \frac{e^r}{2} (1 - e^{-2i\omega t}) \\ &= e^{-i\omega t} (\bar{e}^r \cos \omega t + i e^r \sin \omega t) \end{aligned}$$

$$\Rightarrow \boxed{\Phi(x,t) = \frac{e^{-i\omega t/2}}{\pi^{1/4}} \frac{1}{(\bar{e}^r \cos \omega t + i e^r \sin \omega t)^{1/2}} \times \exp \left\{ -\frac{x^2}{2} \frac{1}{(\bar{e}^r \cos \omega t + i e^r \sin \omega t)^2} \right\}}$$

$$\text{Note } |\Phi(x,t)|^2 = \frac{1}{(2\pi)^2} e^{-\frac{x^2}{2\Delta x^2}}$$

Gaussian whose variance oscillates as

$$\Delta x^2(t) = \frac{1}{2} (\bar{e}^r \cos^2 \omega t + e^r \sin^2 \omega t)$$