Physics 521 Graduate Quantum Mechanics
Diagnostic

(1) A particle is trapped in a 1D infinitely high potential well of width L. Its wave function is given by

$$\psi(x) = \begin{cases} \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right) & |x| \le L/2 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) Show that the wave function is normalized.

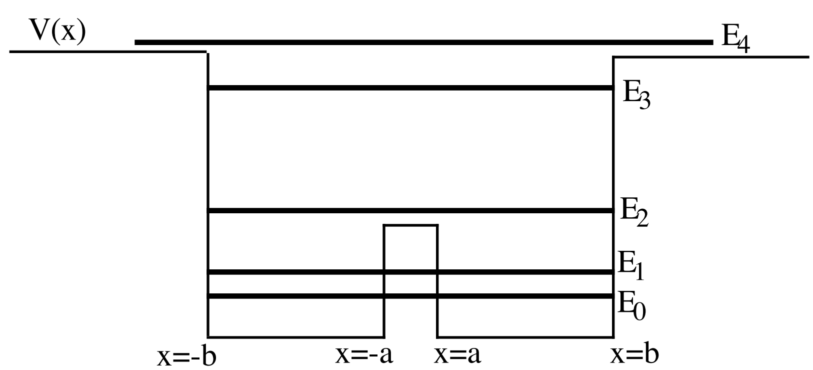
(b) What is the probability of finding the particle in the region $x > 0$?

(c) Find and sketch the momentum space wave function. Interpret your result.

(d) What is the uncertainty in position and momentum associated with the possible measurement outcomes? Is Heisenberg's uncertainty relation satisfied?

(e) What is the expectation value of energy of the system?

(2) The energy levels E_n of a symmetric potential well V(x) are denoted below.



(a) How many bound states are there?

(b) Sketch the wave functions for the first three levels (n=0,1,2). For each, denote the regions where the particle is "classically" forbidden.

(c) Describe the evolution of the wave function if we start in an equal superposition of eigenstates n=0 and n=1.

(3) A particle of mass m is trapped in a perfect harmonic well. The classical oscillation frequency is ω. At t=0 the state vector for the system (*unnormalized*) is

$$|\psi(0)\rangle = |u_0\rangle + \frac{i}{\sqrt{2}}|u_2\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|u_4\rangle,$$ where $|u_n\rangle$ are the energy eigenstates with $n = 0$ labeling the ground state.

(a) What is the wave function (normalized) in position space at *t=0*?

(b) What is the probability of finding the particle in the ground state at *t=0*?

(c) What is the expectation value of energy?

(d) What is the state vector at a later time *t>0*?

(e) What is the probability of finding the particle in the ground state at later time?

(4) Consider a 2D Hilbert space (describing e.g. a spin 1/2 particle). Let us denote abstractly two different orthonormal basis sets for this space: $\{|+\rangle,|-\rangle\}$ and $\{|\uparrow\rangle,|\downarrow\rangle\}$.

The inner products between these states are given by:

$$\langle\uparrow|+\rangle = \langle\uparrow|-\rangle = \frac{1}{\sqrt{2}}, \quad \langle\downarrow|+\rangle = -\langle\downarrow|-\rangle = \frac{1}{\sqrt{2}}.$$

Let $|\psi\rangle = \frac{1}{\sqrt{3}}|+\rangle + \sqrt{\frac{2}{3}}|-\rangle$ be the state of the spin, and consider the operator

$$\hat{A} = i|\uparrow\rangle\langle\downarrow| - i|\downarrow\rangle\langle\uparrow|.$$

(a) Write the state $|\psi\rangle$ as a superposition of the basis vectors $\{|\uparrow\rangle,|\downarrow\rangle\}$.

(b) Write the matrix representation of $\hat{A}$ in the basis $\{|+\rangle,|-\rangle\}$.

(c) Find the state vector $\hat{A}|\psi\rangle$.

(d) Find the eigenvalues and eigenvectors of $\hat{A}$.

(e) Are the eigenvalues real? Are the eigenvectors orthogonal? Explain your finding.

(5) A particle is in a simultaneous eigenstate of the magnitude of angular momentum squared $\hat{L}^2$ and angular momentum along the z-axis $\hat{L}_z$, with $\hat{L}^2|l,m\rangle = \hbar l(l+1)|l,m\rangle$ and $\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$. The angular momentum is purely orbital and $l = 1$.

(a) What are the possible values of m?

(b) What are the coordinate space wave functions for each value of m.

(c) Show that the commutator $\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z$, by first expressing the Cartesian components of the angular momentum operator in terms of position and momentum. Are there any simultaneous eigenstates of $\hat{L}_x$ and $\hat{L}_y$? Explain your answer.

(d) Find the ket $\hat{L}_x|l,m\rangle$.

(6) A hydrogen atom is in the state with principle quantum number $n=1$.

(a) What are the possible values of angular momentum of the atom in this state.

(b) How many degenerate energy levels have this energy (exclude any spin degrees of freedom or relativistic effects)

(c) Consider the state with angular momentum quantum numbers $l=1$, $m=1$. What is the average speed of the electron?

(d) How much energy is necessary to add to the atom to ionize it?