Physics 521 Fall 2014

Problem Set #1 Mathematical Foundations

Cohen-Tannoudji et al. vol. I, Chap. II

Due: Tuesday Sept. 2, 2014 @ 5PM

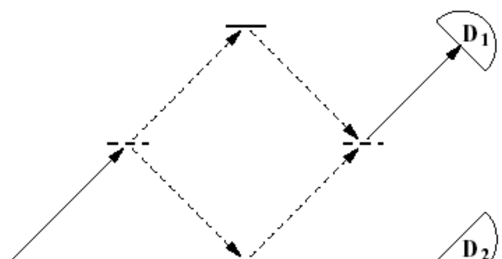
Problem 1: "Interaction-Free" measurement, or: How I learned to stopped worrying about quantum mechanics and love the bomb! (10 points)

Imagine the following scenario (no political statement intended):

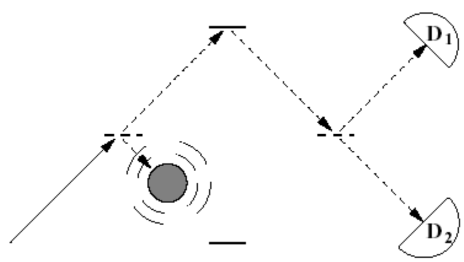
In our current environment of high-tech security, the government has called upon physicists to design the ultimate sensor to detect the presence of bombs without setting them off. Our diabolical foes have designed bombs so sensitive that they absorb photons with unit efficiency, and even a *single* photon will cause them to explode. These bombs are so deadly, that even if we could catch a fraction of them, it would be worthwhile. These bombs do have one defect – some are duds. If the bomb is a dud it is completely transparent.

How can we proceed? If we shine light on the bomb and it is not a dud we are guaranteed to blow it up. Luckily quantum mechanics, with counter-factual reasoning, can come to our rescue, due to an idea by Elitzur and Vaidman.

Consider a perfectly balanced Mach-Zender interferomter so that all photons reach detector D1



Now one of the super bombs (or possible duds) is placed in one arm of the interferometer,



Explain how with 25% probability we can use this interferometer to detect the presence of an active bomb without setting it off? (Hint: What happens if D2 goes "click").

Problem 2: Trace Operation (15 Points)

Let $\hat{A}$ be a linear operator and let $\{|a\rangle\}$ be set of basis vectors (not eigenstates). The *trace of the operator* can then be defined as $\mathrm{Tr}(\hat{A}) = \sum_a \langle a|\hat{A}|a\rangle$

(a) By considering another complete set $\{|b\rangle\}$, show that the trace is independent of basis.

(b) Prove the following properties:

(i) $\mathrm{Tr}\,\hat{A}^\dagger = (\mathrm{Tr}\,\hat{A})^*$

(ii) $\mathrm{Tr}\,(\hat{A}+\hat{B}) = \mathrm{Tr}(\hat{A}) + \mathrm{Tr}\,(\hat{B})$

(iii) $\mathrm{Tr}\,(\hat{A}\hat{B}) = \mathrm{Tr}\,(\hat{B}\hat{A})$

(iv) $Tr(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$

(v) $Tr(|\phi\rangle\langle\psi|\hat{A}) = \langle\psi|\hat{A}|\phi\rangle$

(vi) $\mathrm{Tr}\hat{A} =$ Sum of eigenvalues of $\hat{A}$

(c) The relations (ii) and (iii) imply the vanishing trace of the commutator of two operators. What about $[\hat{x},\hat{p}] = i\hbar$? Are there restrictions on what kinds of operators enjoy the properties in part (b)?

Problem 3: Spin 1/2 operators and eigenstates (15 points)

A spin 1/2 particle is described by a two dimensional Hilbert space. We typically define a "quantization direction" to be the z-direction and define two kets $\{|+_z\rangle, |-_z\rangle\}$ which form an orthonormal basis for the space. The components of the angular momentum operator can then be written

$$\hat{s}_x = \frac{1}{2}|+_z\rangle\langle -_z| + \frac{1}{2}|-_z\rangle\langle +_z|, \quad \hat{s}_y = -\frac{i}{2}|+_z\rangle\langle -_z| + \frac{i}{2}|-_z\rangle\langle +_z|, \quad \hat{s}_z = \frac{1}{2}|+_z\rangle\langle +_z| - \frac{1}{2}|-_z\rangle\langle -_z|.$$

(here I have set $\hbar = 1$)

(a) Find the eigenvalues and eigenvectors (normalized) of $\hat{s}_x, \hat{s}_y, \hat{s}_z$. Are these operators Hermitian and do the eigenvalues/vectors reflect this?

(b) Express $\{|+_z\rangle, |-_z\rangle\}$ in the basis $\{|+_x\rangle, |-_x\rangle\}$, the eigenstates of $\hat{s}_x$. Show that the transformation matrix is unitary.

(c) Express $\hat{s}_x, \hat{s}_y, \hat{s}_z$ as outer products in the basis $\{|+_x\rangle, |-_x\rangle\}$. Please comment on your results.