Physics 521 Fall 2014

Problem Set #2: Due: Tuesday Sept. 9, 2014

Cohen-Tannoudji et al. vol. I, Chap. III

Problem 1: Functions of an operator (20 points)

Quite often it is useful to define a function of an operator. We can do so using Taylor's formula. Consider a function $f(z)$ on the real numbers which is *analytic* at x, so that

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \text{ where } f_n = \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=0} ,$$

converges. Given an operator $\hat{A}$, we define

$$f(\hat{A}) = \sum_{n=0}^{\infty} f_n \hat{A}^n .$$

(a) Show that if $\hat{A}$ is normal operator with eigen-decomposition $\hat{A} = \sum_a a\, |a\rangle\langle a|$,

$$f(\hat{A}) = \sum_{n=0}^{\infty} f(a) |a\rangle\langle a| .$$

(Note to the purists, we must "analytically continue" $f(x)$ into the complex plane).

(b) An important example is the function of an anti-Hermitian operator $\hat{A}$, $f(\hat{A}) = e^{\hat{A}}$. Show that this operator is *unitary*.

(c) Writing $\hat{A} = i\hat{H}$ where $\hat{H}$ is Hermitian, express this operator in terms of the eigenvalues and eigenvectors of $\hat{H}$.

(d) Consider $\hat{U} = e^{\hat{A}}$, where $\hat{A}$ is anti-Hermitian. Show that

$$\hat{U}\hat{B}\hat{U}^{\dagger} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]] + \ldots$$

This is known as the Baker-Cambell-Hausdorff lemma and we will be using it extensively.

Problem 2: The polar decomposition (15 Points)

We saw in class that any operator $\hat{M}$ can be decomposed as $\hat{M} = \hat{H} + \hat{A}$, where $\hat{H} = \left(\hat{M} + \hat{M}^\dagger\right)/2$ is a Hermitian operator and $\hat{A} = \left(\hat{M} - \hat{M}^\dagger\right)/2$ is an anti-Hermitian operator. This is the operator equivalent of a decomposition of a complex number into a sum of a real and imaginary numbers. Complex numbers also have a "polar decomposition", $z = re^{i\theta}$, where $r = |z| = \sqrt{zz^*} \geq 0$ is the magnitude and $e^{i\theta} = z/\sqrt{zz^*}$, where $\phi = \arg(z)$ is the phase. This problem explores the generalization to operators on a complex vector space.

Consider an invertible operator $\hat{M}$ acting on a complex vector space with finite dimension. Define $\hat{R}_L = \sqrt{\hat{M}\hat{M}^\dagger}$ and $\hat{R}_R = \sqrt{\hat{M}^\dagger \hat{M}}$ as the "left" and "right" magnitude-operators, respectively. They are "positive operators" (their eigenvalues are ≥ 0).

(a) Show that $\hat{M} = \hat{R}_L \hat{U} = \hat{U}\hat{R}_R$, where $\hat{U}$ is a unique unitary operator. This decomposition is known as the "polar decomposition" and is useful in many contexts. Explain why this term makes sense as a generalization from complex numbers.

Note: This decomposition holds in finite dimensions even if $\hat{M}$ is not invertible, but in that case $\hat{U}$ is not unique.

(b) A normal operator has an eigen-decomposition $\hat{M} = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$, where λ are complex numbers and $\{|\lambda\rangle\}$ are the eigenvectors. Write the Hermitian/anti-Hermitian and polar decompositions for this operator.

(c) Find the polar decomposition of the operator $\hat{M} = \hat{1} + (\hat{\sigma}_x - i\hat{\sigma}_y)/2$.

In infinite dimensions, the polar decomposition is more tricky (as many things are). Generally it does not work as we will see when we study the harmonic oscillator. Keep this in mind!

Problem 3: The angular momentum operator. (20 points)

The orbital angular momentum operator for a particle with momentum $\hat{\mathbf{p}}$ and position $\hat{\mathbf{x}}$ is

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}, \text{ or in component form } \hat{L}_i = \sum_{j,k} \varepsilon_{ijk} \hat{x}_j \hat{p}_k,$$

where i,j,k index the Cartesian components and sums go from 1 to 3 (1=x, 2=y,3=z).

(a) We know the famous "canonical commutation relations" $\left[\hat{x}_j, \hat{p}_k\right] = i\delta_{jk}\hbar$ (position and momentum for different Cartesian coordinates, otherwise not).

Prove the following standard SU(2) commutation relations: $[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k$

Note, As we saw, this holds for spin angular mometum (e.g. spin-1/2):

$$[\hat{s}_i, \hat{s}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{s}_k$$

(b) Further show some, perhaps less familiar, commutators

$$[\hat{L}_i, \hat{x}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{x}_k, \; [\hat{L}_i, \hat{p}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{p}_k, \; [\hat{L}_i, \hat{r}^2] = [\hat{L}_i, \hat{p}^2] = [\hat{L}_i, \hat{L}^2] = 0.$$

(c) Prove the uncertainty principle for angular momentum $\Delta J_x \Delta J_y \geq \frac{\hbar}{2}\left|\left\langle \hat{J}_z \right\rangle\right|$, where $\hat{J}_i$ is component of generic angular momentum, orbital or spin.

(d) Given a spin-1/2 in the state $\left|+_z\right\rangle$, find Δs_x and Δs_y. Show that the uncertainty principle for angular momentum is satisfied.