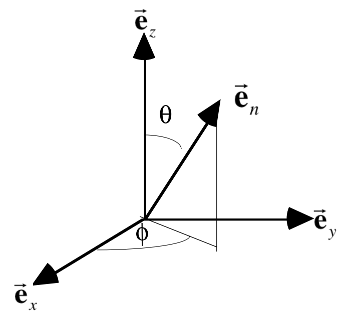
**Physics 521 Fall 2014**

**Problem Set #3: Due: Tuesday Sept. 16, 2014**

**Problem 1: Measurements on a two-state system** (15 points)

Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis $z$,




we can define the ket $\left|+_n\right\rangle = \cos(\theta/2)\left|+_z\right\rangle + e^{i\phi}\sin(\theta/2)\left|-_z\right\rangle$, as the state with spin $+\hbar/2$ along the axis $\vec{\mathbf{e}}_n$.

**(a)** Show that $\hat{\mathbf{s}}\cdot\vec{\mathbf{e}}_n\left|+_n\right\rangle = \frac{\hbar}{2}\left|+_n\right\rangle$, where $\hat{\mathbf{s}} = \hat{s}_x\,\vec{\mathbf{e}}_x + \hat{s}_y\,\vec{\mathbf{e}}_y + \hat{s}_z\,\vec{\mathbf{e}}_z$, with $\left\{\hat{s}_x, \hat{s}_y, \hat{s}_z\right\}$ the three components of the spin 1/2 operator from Problem Set #1.

Now consider a beam of spin 1/2 atoms that goes through a series of Stern-Gerlach-type measurements as follows:

(i) The first measurement accepts $s_z = +\hbar/2$ and rejects $s_z = -\hbar/2$.

(ii) The second measurement accepts $s_n = +\hbar/2$ and rejects $s_n = -\hbar/2$ (along axis $\vec{\mathbf{e}}_n$).

(iii) The third measurement accepts $s_z = -\hbar/2$ and rejects $s_z = +\hbar/2$.

**(b)** What is the probability of detecting the final spin with $s_z = -\hbar/2$ given an atom which passes through the first apparatus?

**(c)** How must we orient the second apparatus if we are to maximize this probability. Please *interpret* your result.

**Problem 2: Beyond Projective Measurement** (30 points)

(a) Bayes rules with Gaussians.
Let's consider a *classical* problem (no quantum uncertainty). Suppose we're trying to measure the position of a particle and we assign a prior probability distribution, $p(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-(x-x_0)^2/2\sigma_0^2\right]$. Our measuring device is not perfect. Due to noise is can only measure with a resolution $\Delta$. That is, when I measure the position, I must put error bars on this. Thus, if my detector registers the position as *y*, I assign likelihood that the position was *x* to a Gaussian, $p(y\,|\,x) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left[-(y-x)^2/2\Delta^2\right]$.

Use Bayes theorem to show that, given the new data, I must now update my probability assignment of the position to a new Gaussian,

$$p(x\,|\,y) = \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left[-(x-x')^2/2\sigma'^2\right],$$

Where $x' = x_0 + K_1(y - x_0)$, $\sigma'^2 = K_2\sigma_0^2$, with $K_1 = \frac{\sigma_0^2}{\sigma_0^2 + \Delta^2}$ $\quad K_2 = \frac{\Delta^2}{\sigma_0^2 + \Delta^2}$.

Comment on the behavior as the measurement resolution improves.

(b) POVM for spin projection. Consider the spin *J*, with eigenvectors of $\hat{J}_z$, $\hat{J}_z|M\rangle = M|M\rangle$. Let up define a set of operators know a "Kraus operators," according to

$$\hat{A}_\mu \equiv \sum_{M=-J}^{J} \frac{1}{\left(2\pi\Delta^2\right)^{1/4}} e^{-\frac{(\mu-M)^2}{4\Delta^2}} |M\rangle\langle M| \text{ , where, } -\infty < \mu < \infty \text{ is a continuous variable}$$

Show that $\left\{\hat{E}_\mu = \hat{A}_\mu^\dagger \hat{A}_\mu\right\}$ is a POVM, i.e., $\hat{E}_\mu \geq 0$ and $\int_{-\infty}^{+\infty} d\mu\, \hat{E}_\mu = \hat{1}$.

(c) Comment on the POVM elements $\left\{\hat{E}_\mu\right\}$ as the resolution becomes perfect $\Delta \to 0$.

(d) Suppose the state before a measurement is $|\psi_<\rangle = \sum_{M=-J}^{J} \frac{1}{\left(2\pi\sigma_0^2\right)^{1/4}} e^{-\frac{(M-M_0)^2}{4\sigma_0^2}} |M\rangle$

(i) What is $\langle \hat{J}_z \rangle$ and $\Delta J_z$ ? (assume *J*>>1).

(ii) What is the probability $p(\mu)$ of finding outcome $\mu$ corresponding to $\hat{E}_\mu$

(e) Conditioned on finding $\mu$, corresponding to $\hat{E}_\mu$, the post-measurement state is updated according to a generalization of the projection postulate, $|\psi_>\rangle = \frac{\hat{A}_\mu|\psi_<\rangle}{\| \hat{A}_\mu|\psi_<\rangle \|}$.

(i) Find $|\psi_>\rangle$, $\langle \hat{J}_z \rangle$ and $\Delta J_z$ after the measurement. Comment in relation to part (a)

(ii) Discuss two limiting cases: $\Delta << \sigma_0$ and $\sigma_0 << \Delta$.