Physics 521 Fall 2014

Problem Set #4: Due: Tuesday Sept. 23, 2014

Problem 1: Some Algebra With Density Matrices (25 points)

Suppose we have an ensemble of spin 1/2 particles that consists of a statistical mixture with 1/3 of the ensemble in the state $|+_z\rangle$ and 2/3 of the ensemble in the state $|-_z\rangle$.

(a) Find the matrix of the density operator in the basis $\{|+_z\rangle, |-_z\rangle\}$, and in the basis of eigenstates of $\hat{S}_x$, $\{|+_x\rangle, |-_x\rangle\}$

(b) Find, $\langle \hat{S}_x \rangle, \langle \hat{S}_y \rangle, \langle \hat{S}_z \rangle$ for this state, and compare it to the result for the mixed state $\hat{\rho} = \frac{1}{2}|+_z\rangle\langle +_z| + \frac{1}{2}|-_z\rangle\langle -_z|$. Please comment on your findings.

(c) Find $\langle \hat{\mathbf{S}} \cdot \mathbf{e}_n \rangle$ for an arbitrary axis $\mathbf{e}_n$, for the 1/3 - 2/3 mixture and for the completely mixed state with a 1/2 - 1/2 mixture of $\{|+_z\rangle, |-_z\rangle\}$,. Please comment.

Consider now another state: a statistical mixture with 1/2 of the ensemble in the state $|+_z\rangle$ and 1/2 of the ensemble in the state $|+_x\rangle$.

(d) What is the *purity* of this state? Is it completely mixed? Comment.

(e) What are the eigenvalues and eigenvectors of this state? Express the density matrix as a statistical mixture of its eigenstates.

Problem 2: From Cohen-Tannoudji vol. I, Exercise 14, page 347. (25 points).

Consider a physical system whose state space, which is three dimensional, spanned by the orthonormal basis formed by the three kets $|u_1\rangle, |u_2\rangle, |u_3\rangle$. In this basis, the Hamiltonian operator of the system and the two observables $\hat{A}$ and $\hat{\mathrm{B}}$ have the representations:

$$\hat{H} \doteq \hbar\omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad \hat{A} \doteq a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad \hat{B} \doteq b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ω_0, a and b are positive real constants.

The state at time t=0 is: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$.

(a) The energy of the system is measured (projectively) at t=0. What values can be found and with what probabilities? What is the mean value and rms uncertainty in energy?

(b) If one measures observable A at t=0 (projectively), what values can be found and with what probabilities? What is the state vector immediately after the measurement?

(c) Find $|\psi(t)\rangle$.

(d) Calculate $\langle \hat{A} \rangle(t)$ and $\langle \hat{B} \rangle(t)$, for time t>0. What comments can be made?

(e) What results can be obtained if observable A is measured (projectively) at time t? Repeat for B. Interpret.